$$9k^2-4<0$$
$$(3k-2)(3k+2)<0$$
$$\begin{cases}3k-2>0\\3k+2<0\end{cases} \quad \text{or} \quad \begin{cases}3k-2<0\\3k+2>0\end{cases}$$
no solution or $-\frac{2}{3}<k<\frac{2}{3}$

$$-\frac{2}{3}<k<\frac{2}{3}$$

CHAPTER 3

Exercise 3A (p.71)

1. Let $P(n)$ be the proposition

"$1+4+7+\cdots+(3n-2)=\frac{n}{2}(3n-1)$".

When $n=1$, L.H.S. $=1$

R.H.S. $=\frac{1}{2}[3(1)-1]=1$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $1+4+7+\cdots+(3k-2)=\frac{k}{2}(3k-1)$

Then $1+4+7+\cdots+(3k-2)+[3(k+1)-2]$

$=\frac{k}{2}(3k-1)+(3k+1)$

$=\frac{3k^2}{2}-\frac{k}{2}+3k+1$

$=\frac{3k^2-k+6k+2}{2}$

$=\frac{3k^2+5k+2}{2}$

$=\frac{(k+1)(3k+2)}{2}$

$=\frac{k+1}{2}[3(k+1)-1]$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

2. Let $P(n)$ be the proposition

"$2^3+4^3+6^3+\cdots+(2n)^3=2n^2(n+1)^2$".

When $n=1$, L.H.S. $=2^3=8$

R.H.S. $=2(1)^2(1+1)^2=8$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $2^3+4^3+6^3+\cdots+(2k)^3=2k^2(k+1)^2$

Then $2^3+4^3+\cdots+(2k)^3+[2(k+1)]^3$

$=2k^2(k+1)^2+8(k+1)^3$

$=2(k+1)^2(k^2+4k+4)$

$=2(k+1)^2(k+2)^2$

$=2(k+1)^2[(k+1)+1]^2$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

3. Let $P(n)$ be the proposition

"$2^2+4^2+6^2+\cdots+(2n)^2=\frac{2}{3}n(n+1)(2n+1)$"

When $n=1$, L.H.S. $=2^2=4$

R.H.S. $=\frac{2}{3}(1)(1+1)(2+1)=4$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $2^2+4^2+6^2+\cdots+(2k)^2=\frac{2}{3}k(k+1)(2k+1$

Then $2^2+4^2+6^2+\cdots+(2k)^2+[2(k+1)]^2$

$=\frac{2}{3}k(k+1)(2k+1)+4(k+1)^2$

$=\frac{2}{3}(k+1)[k(2k+1)+6(k+1)]$

$=\frac{2}{3}(k+1)(2k^2+k+6k+6)$

$=\frac{2}{3}(k+1)(2k^2+7k+6)$

$=\frac{2}{3}(k+1)(k+2)(2k+3)$

$=\frac{2}{3}(k+1)[(k+1)+1][2(k+1)+1]$

Thus assuming $P(k)$ is true for any positive integ k, $P(k+1)$ is also true. By the principle mathematical induction, $P(n)$ is true for a positive integers n.

4. Let $P(n)$ be the proposition

"$a+(a+d)+(a+2d)+\cdots+[a+(n-1)d]$

$=\frac{n}{2}[2a+(n-1)d]$".

When $n=1$, L.H.S. $=a$

R.H.S. $=\frac{1}{2}[2a+(1-1)d]=a$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $a+(a+d)+(a+2d)+\cdots+[a+(k-1)d]$

$=\frac{k}{2}[2a+(k-1)d]$

Then

$a+(a+d)+\cdots+[a+(k-1)d]+\{a+[(k+1)-1]d$

$=\frac{k}{2}(2a+(k-1)d]+(a+kd)$

$=\frac{1}{2}[2ka+k(k-1)d+2a+2kd]$

$=\frac{1}{2}[(k+1)2a+k^2d-kd+2kd]$

$=\frac{1}{2}[(k+1)2a+k(k+1)d]$

$=\frac{1}{2}(k+1)(2a+kd)$

$=\frac{k+1}{2}\{2a+[(k+1)-1]d\}$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

5. Let $P(n)$ be the proposition
"$2\cdot 3+4\cdot 6+6\cdot 9+\cdots+(2n)(3n) = n(n+1)(2n+1)$".

When $n=1$, L.H.S. $=(2)(3)=6$

R.H.S. $=(1)(1+1)(2+1)=6$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $2\cdot 3+4\cdot 6+6\cdot 9+\cdots+(2k)(3k) = k(k+1)(2k+1)$

Then $2\cdot 3+4\cdot 6+6\cdot 9+\cdots+(2k)(3k)+[2(k+1)][3(k+1)]$
$= k(k+1)(2k+1)+6(k+1)^2$
$=(k+1)(2k^2+k+6k+6)$
$=(k+1)(2k^2+7k+6)$
$=(k+1)(k+2)(2k+3)$
$=(k+1)[(k+1)+1][2(k+1)+1]$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

6. Let $P(n)$ be the proposition
"$1\cdot 4+2\cdot 7+3\cdot 10+\cdots+n(3n+1)=n(n+1)^2$".

When $n=1$, L.H.S. $=1\cdot 4=4$

R.H.S. $=(1+1)^2=4$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $1\cdot 4+2\cdot 7+3\cdot 10+\cdots+k(3k+1)=k(k+1)^2$

Then
$1\cdot 4+2\cdot 7+\cdots+k(3k+1)+(k+1)+[3(k+1)+1]$
$=k(k+1)^2+(k+1)(3k+4)$
$=(k+1)(k^2+k+3k+4)$
$=(k+1)(k^2+4k+4)$
$=(k+1)(k+2)^2$
$=(k+1)[(k+1)+1]^2$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

7. Let $P(n)$ be the proposition
"$1\cdot 3+2\cdot 3^2+3\cdot 3^3+\cdots+n\cdot 3^n = \dfrac{(2n-1)3^{n+1}+3}{4}$".

When $n=1$, L.H.S. $=1\cdot 3=3$

R.H.S. $=\dfrac{(2-1)3^{1+1}+3}{4}=3$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $1\cdot 3+2\cdot 3^2+3\cdot 3^3+\cdots+k\cdot 3^k = \dfrac{(2k-1)3^{k+1}+3}{4}$

Then $1\cdot 3+2\cdot 3^2+3\cdot 3^3+\cdots+k\cdot 3^k+(k+1)\cdot 3^{k+1}$

$$=\frac{(2k-1)3^{k+1}+3}{4}+(k+1)\cdot 3^{k+1}$$
$$=\frac{1}{4}[(2k-1)3^{k+1}+3+4(k+1)\cdot 3^{k+1}]$$
$$=\frac{(2k-1+4k+4)3^{k+1}+3}{4}$$
$$=\frac{(6k+3)\cdot 3^{k+1}+3}{4}$$
$$=\frac{(2k+1)\cdot 3^{k+2}+3}{4}$$
$$=\frac{[2(k+1)-1]3^{(k+1)+1}+3}{4}$$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

8. Let $P(n)$ be the proposition
"$\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$".

When $n=1$, L.H.S. $=\dfrac{1}{1\cdot 2}=\dfrac{1}{2}$

R.H.S. $=\dfrac{1}{1+1}=\dfrac{1}{2}$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{k(k+1)}=\dfrac{k}{k+1}$

Then

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)[(k+1)+1]}$$
$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$
$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$
$$=\frac{(k+1)^2}{(k+1)(k+2)}$$
$$=\frac{k+1}{k+2}$$
$$=\frac{k+1}{(k+1)+1}$$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

9. Let $P(n)$ be the proposition
"$\dfrac{1}{2\cdot 5}+\dfrac{1}{5\cdot 8}+\dfrac{1}{8\cdot 11}+\cdots+\dfrac{1}{(3n-1)(3n+2)} = \dfrac{n}{6n+4}$".

When $n=1$, L.H.S. $=\dfrac{1}{2\cdot 5}=\dfrac{1}{10}$

R.H.S. $=\dfrac{1}{6+4}=\dfrac{1}{10}$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $\dfrac{1}{2\cdot 5}+\dfrac{1}{5\cdot 8}+\dfrac{1}{8\cdot 11}+\cdots+\dfrac{1}{(3k-1)(3k+2)} = \dfrac{k}{6k+4}$

Then $\dfrac{1}{2\cdot 5}+\dfrac{1}{5\cdot 8}+\dfrac{1}{8\cdot 11}+\cdots+\dfrac{1}{(3k-1)(3k+2)}$

$$+\frac{1}{[3(k+1)-1][3(k+1)+2]}$$
$$=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$$
$$=\frac{k(3k+5)+2}{2(3k+2)(3k+5)}$$
$$=\frac{3k^2+5k+2}{2(3k+2)(3k+5)}$$
$$=\frac{(3k+2)(k+1)}{2(3k+2)(3k+5)}$$
$$=\frac{k+1}{2(3k+5)}$$
$$=\frac{k+1}{6(k+1)+4}$$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

10. Let $P(n)$ be the proposition
"$\dfrac{1}{1\cdot 3\cdot 5}+\dfrac{1}{3\cdot 5\cdot 7}+\dfrac{1}{5\cdot 7\cdot 9}+\cdots+\dfrac{1}{(2n-1)(2n+1)(2n+3)}=\dfrac{n(n+2)}{3(2n+1)(2n+3)}$".

When $n=1$, L.H.S. $=\dfrac{1}{1\cdot 3\cdot 5}=\dfrac{1}{15}$

R.H.S. $=\dfrac{(1+2)}{3(2+1)(2+3)}=\dfrac{1}{15}$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $\dfrac{1}{1\cdot 3\cdot 5}+\dfrac{1}{3\cdot 5\cdot 7}+\dfrac{1}{5\cdot 7\cdot 9}+\cdots+\dfrac{1}{(2k-1)(2k+1)(2k+3)}=\dfrac{k(k+2)}{3(2k+1)(2k+3)}$

Then

$$\frac{1}{1\cdot 3\cdot 5}+\frac{1}{3\cdot 5\cdot 7}+\frac{1}{5\cdot 7\cdot 9}+\cdots+\frac{1}{(2k-1)(2k+1)(2k+3)}$$
$$+\frac{1}{[2(k+1)-1][2(k+1)+1][2(k+1)+3]}$$
$$=\frac{k(k+2)}{3(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)(2k+5)}$$
$$=\frac{k(k+2)(2k+5)+3}{3(2k+1)(2k+3)(2k+5)}$$
$$=\frac{2k^3+9k^2+10k+3}{3(2k+1)(2k+3)(2k+5)}$$
$$=\frac{(k+1)(2k^2+7k+3)}{3(2k+1)(2k+3)(2k+5)}$$
$$=\frac{(k+1)(2k+1)(k+3)}{3(2k+1)(2k+3)(2k+5)}$$
$$=\frac{(k+1)(k+3)}{3(2k+3)(2k+5)}$$
$$=\frac{(k+1)[(k+1)+2]}{3[2(k+1)+1][(2(k+1)+3]}$$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

11. Let $P(n)$ be the proposition
"$\dfrac{1}{1\times 4}+\dfrac{1}{4\times 7}+\dfrac{1}{7\times 10}+\cdots+\dfrac{1}{(3n-5)(3n-2)} = \dfrac{n-1}{3n-2}$".

When $n=2$, L.H.S. $=\dfrac{1}{1\times 4}=\dfrac{1}{4}$

R.H.S. $=\dfrac{2-1}{3\times 2-2}=\dfrac{1}{4}$

$\therefore$ $P(2)$ is true.

Assume $P(k)$ is true for any positive integer $k\ge 2$

i.e. $\dfrac{1}{1\times 4}+\dfrac{1}{4\times 7}+\dfrac{1}{7\times 10}+\cdots+\dfrac{1}{(3k-5)(3k-2)} = \dfrac{k-1}{3k-2}$

Then

$$\frac{1}{1\times4}+\frac{1}{4\times7}+\frac{1}{7\times10}+\cdots+\frac{1}{(3k-5)(3k-2)}$$
$$+\frac{1}{[3(k+1)-5][3(k+1)-2]}$$
$$=\frac{k-1}{3k-2}+\frac{1}{(3k-2)(3k+1)}$$
$$=\frac{1}{3k-2}[\frac{(k-1)(3k+1)+1}{3k+1}]$$
$$=\frac{1}{3k-2}\cdot\frac{k(3k-2)}{3k+1}$$
$$=\frac{k}{3k+1}$$
$$=\frac{(k+1)-1}{3(k+1)-2}$$

Thus assuming $P(k)$ is true for any positive integer $k\geq2$, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers $n\geq2$.

12. Let $P(n)$ be the proposition

"$(1-\frac{4}{9})(1-\frac{4}{25})\cdots[1-\frac{4}{(2n-1)^2}]=\frac{2n+1}{3(2n-1)}$".

When $n=2$, L.H.S. $=1-\frac{4}{9}=\frac{5}{9}$

R.H.S. $=\frac{2(2)+1}{3[2(2)-1]}=\frac{5}{9}$

$\therefore$ $P(2)$ is true.

Assume $P(k)$ is true for any positive integer $k\geq2$.

i.e. $(1-\frac{4}{9})(1-\frac{4}{25})\cdots[1-\frac{4}{(2k-1)^2}]=\frac{2k+1}{3(2k-1)}$

Then

$$(1-\frac{4}{9})(1-\frac{4}{25})\cdots[1-\frac{4}{(2k-1)^2}]\{1-\frac{4}{[2(k+1)-1]^2}\}$$
$$=\frac{2k+1}{3(2k-1)}(1-\frac{4}{(2k+1)^2})$$
$$=\frac{(2k+1)[(2k+1)^2-4]}{3(2k-1)(2k+1)^2}$$
$$=\frac{(2k-1)(2k+3)}{3(2k-1)(2k+1)}$$
$$=\frac{2k+3}{3(2k+1)}$$
$$=\frac{2(k+1)+1}{3[2(k+1)-1)]}$$

Thus assuming $P(k)$ is true for any positive integer $k\geq2$, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers $n\geq2$.

13. Let $P(n)$ be the proposition

"$1\times1!+2\times2!+\cdots+n\times n!=(n+1)!-1$".

When $n=1$, L.H.S. $=1\times1!=1$

R.H.S. $=(1+1)!-1=1$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $1\times1!+2\times2!+\cdots+k\times k!=(k+1)!-1$

Then $1\times1!+2\times2!+\cdots+k\times k!+(k+1)\times(k+1)!$
$$=(k+1)!-1+(k+1)\times(k+1)!$$
$$=(k+1)!(1+k+1)-1$$
$$=(k+1)!(k+2)-1$$
$$=(k+2)!-1$$
$$=[(k+1)+1]!-1$$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

14. No solution is provided for the H.K.C.E.E. question because of the copyright reasons.

15. (a) Let $P(n)$ be the proposition

"$1\cdot3^2+2\cdot5^2+3\cdot7^2+\cdots+n(2n+1)^2$
$=\frac{1}{6}n(n+1)(6n^2+14n+7)$".

When $n=1$, L.H.S. $=1\cdot3^2=9$

R.H.S. $=\frac{1}{6}(1+1)(6+14+7)=9$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $1\cdot3^2+2\cdot5^2+3\cdot7^2+\cdots+k(2k+1)^2$
$=\frac{1}{6}k(k+1)(6k^2+14k+7)$

Then

$$1\cdot3^2+2\cdot5^2+3\cdot7^2+\cdots+k(2k+1)^2$$
$$+(k+1)[2(k+1)+1]^2$$
$$=\frac{1}{6}k(k+1)(6k^2+14k+7)+(k+1)(2k+3)^2$$
$$=\frac{1}{6}(k+1)[k(6k^2+14k+7)+6(2k+3)^2]$$
$$=\frac{1}{6}(k+1)(6k^3+14k^2+7k+24k^2+72k+54)$$
$$=\frac{1}{6}(k+1)(6k^3+38k^2+79k+54)$$
$$=\frac{1}{6}(k+1)(k+2)(6k^2+26k+27)$$
$$=\frac{1}{6}(k+1)(k+2)(6k^2+12k+6+14k+14+7)$$
$$=\frac{1}{6}(k+1)[(k+1)+1][6(k+1)^2+14(k+1)+7]$$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

(b) $11\cdot23^2+12\cdot25^2+13\cdot27^2+\cdots+20\cdot41^2$
$$=\frac{1}{6}(20)(21)(2\,400+280+7)$$
$$-\frac{1}{6}(10)(11)(600+140+7)\text{ (By (a))}$$
$$=188\,090-13\,695$$
$$=\underline{\underline{174\,395}}$$

16. (a) Let $P(n)$ be the proposition

"$1^2\cdot4+2^2\cdot5+3^2\cdot6+\cdots+n^2(n+3)$
$=\frac{1}{4}n(n+1)(n^2+5n+2)$".

When $n=1$, L.H.S. $=1^2\cdot4=4$

R.H.S. $=\frac{1}{4}(1+1)(1+5+2)=4$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $1^2\cdot4+2^2\cdot5+3^2\cdot6+\cdots+k^2(k+3)$
$=\frac{1}{4}k(k+1)(k^2+5k+2)$

Then

$$1^2\cdot4+2^2\cdot5+3^2\cdot6+\cdots+k^2(k+3)$$
$$+(k+1)^2[(k+1)+3]$$
$$=\frac{1}{4}k(k+1)(k^2+5k+2)+(k+1)^2(k+4)$$
$$=\frac{1}{4}(k+1)(k^3+5k^2+2k+4k^2+20k+16)$$
$$=\frac{1}{4}(k+1)(k^3+9k^2+22k+16)$$
$$=\frac{1}{4}(k+1)(k+2)(k^2+7k+8)$$
$$=\frac{1}{4}(k+1)(k+2)(k^2+2k+1+5k+5+2)$$
$$=\frac{1}{4}(k+1)[(k+1)+1][(k+1)^2+5(k+1)+2]$$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

(b) $1^2\cdot2+2^2\cdot3+\cdots+n^2(n+1)$
$$=1^2(4-2)+2^2(5-2)+3^2(6-2)$$
$$+\cdots+n^2[(n+3)-2]$$
$$=1^2\cdot4-1^2\cdot2+2^2\cdot5-2^2\cdot2+3^2\cdot6-3^2\cdot2$$
$$+\cdots+n^2(n+3)-n^2\cdot2$$

$$=[1^2\cdot4+2^2\cdot5+3^2\cdot6+\cdots+n^2(n+3)]$$
$$-2(1^2+2^2+\cdots+n^2)$$
$$=\frac{1}{4}n(n+1)(n^2+5n+2)-2\cdot\frac{1}{6}n(n+1)(2n+$$
$$=\frac{1}{4}n(n+1)(n^2+5n+2)-\frac{1}{3}n(n+1)(2n+$$
$$=\frac{1}{12}n(n+1)(3n^2+15n+6-8n-4)$$
$$=\frac{1}{12}n(n+1)(3n^2+7n+2)$$
$$=\underline{\underline{\frac{1}{12}n(n+1)(n+2)(3n+1)}}$$

17. (a) Let $P(n)$ be the proposition

"$1\cdot2+2\cdot3+3\cdot4+\cdots+n(n+1)$
$=\frac{1}{3}n(n+1)(n+2)$".

When $n=1$, L.H.S. $=1\cdot2=2$

R.H.S. $=\frac{1}{3}(1+1)(1+2)=2$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer

i.e. $1\cdot2+2\cdot3+3\cdot4+\cdots+k(k+1)$
$=\frac{1}{3}k(k+1)(k+2)$

Then

$$1\cdot2+2\cdot3+\cdots+k(k+1)+(k+1)[(k+1)+1]$$
$$=\frac{1}{3}k(k+1)(k+2)+(k+1)(k+2)$$
$$=\frac{1}{3}(k+1)(k+2)(k+3)$$
$$=\frac{1}{3}(k+1)[(k+1)+1][(k+1)+2)$$

Thus assuming $P(k)$ is true for any positi integer k, $P(k+1)$ is also true. By th principle of mathematical induction, $P(n)$ true for all positive integers n.

(b) (i) $51\cdot52+52\cdot53+\cdots+100\cdot101$
$$=1\cdot2+2\cdot3+\cdots+100\cdot101$$
$$-(1\cdot2+2\cdot3+\cdots+50\cdot51)$$
$$=\frac{1}{3}(100)(101)(102)-\frac{1}{3}(50)(51)(52)$$
$$=343\,400-44\,200$$
$$=\underline{\underline{299\,200}}$$

(ii) The nth term of the series
$$=1+2+\cdots+n$$
$$=\frac{1}{2}n(n+1)$$

Sum of the first n term:

$=\frac{1}{2}(1)(2)+\frac{1}{2}(2)(3)+\cdots+\frac{1}{2}n(n+1)$

$=\frac{1}{2}[(1)(2)+(2)(3)+\cdots+n(n+1)]$

$=\frac{1}{2}\cdot\frac{1}{3}n(n+1)(n+2)$

$=\frac{1}{6}n(n+1)(n+2)$

$\therefore\ 1+(1+2)+\cdots+(1+2+\cdots+100)$

$=\frac{1}{6}(100)(101)(102)$

$=\underline{\underline{171\ 700}}$

18. (a) Let $P(n)$ be the proposition

"$1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$".

When $n=1$, L.H.S. $=1^2=1$

R.H.S. $=\frac{1}{6}(1+1)(2+1)=1$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $1^2+2^2+3^2+\cdots+k^2$

$=\frac{1}{6}k(k+1)(2k+1)$

Then

$1^2+2^2+\cdots+k^2+(k+1)^2$

$=\frac{1}{6}k(k+1)(2k+1)+(k+1)^2$

$=\frac{1}{6}(k+1)(2k^2+k+6k+6)$

$=\frac{1}{6}(k+1)(2k^2+7k+6)$

$=\frac{1}{6}(k+1)(k+2)(2k+3)$

$=\frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

(b) (i) $2^2+4^2+6^2+\cdots+(2n)^2$

$=4(1^2+2^2+3^2+\cdots+n^2)$

$=4\cdot\frac{1}{6}n(n+1)(2n+1)$

$=\underline{\underline{\frac{2}{3}n(n+1)(2n+1)}}$

(ii) $1^2-2^2+3^2-4^2+\cdots-40^2$

$=1^2+2^2+\cdots+40^2$

$\quad-2(2^2+4^2+\cdots+40^2)$

$=\frac{1}{6}(40)(41)(81)-2\cdot\frac{2}{3}(20)(21)(41)$

$=22\ 140-22\ 960$

$=\underline{\underline{-820}}$

(c) $1\cdot1+2\cdot3+3\cdot5+\cdots+n(2n-1)$

$=1\cdot(2\cdot1-1)+2\cdot(2\cdot2-1)$

$\quad+3(2\cdot3-1)+\cdots+n(2n-1)$

$=2\cdot1^2-1+2\cdot2^2-2+2\cdot3^2$

$\quad-3+\cdots+2n^2-n$

$=2(1^2+2^2+3^2+\cdots+n^2)$

$\quad-(1+2+3+\cdots+n)$

$=2\cdot\frac{1}{6}n(n+1)(2n+1)-\frac{1}{2}n(n+1)$ (By **(a)**)

$=\frac{1}{3}n(n+1)(2n+1)-\frac{1}{2}n(n+1)$

$=\frac{1}{6}n(n+1)(4n+2-3)$

$=\underline{\underline{\frac{1}{6}n(n+1)(4n-1)}}$

Exercise 3B (p.75)

1. Let $P(n)$ be the proposition "9^n-2^n is divisible by 7".

When $n=1$, $9-2=7$ which is divisible by 7.

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $9^k-2^k=7N$ where N is an integer.

Then $9^{k+1}-2^{k+1}$

$=9\cdot9^k-2\cdot2^k$

$=9(7n+2^k)-2\cdot2^k$

$=9(7N)+7\cdot2^k$

$=7(9N+2^k)$

which is divisible by 7.

Hence $P(k+1)$ is true if $P(k)$ is true for any positive integer k. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

2. Let $P(n)$ be the proposition "$2^{2n}-1$ is divisible by 3".

When $n=1$, $2^2-1=3$ which is divisible by 3.

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $2^{2k}-1=3N$ where N is an integer.

Then $2^{2(k+1)}-1$

$=2^{2k}\cdot2^2-1$

$=4(3N+1)-1$

$=4(3N)+3$

$=3(4N+1)$

which is divisible by 3.

Hence $P(k+1)$ is true if $P(k)$ is true for any positive integer k. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

3. Let $P(n)$ be the proposition "9^n-4^n is divisible by 5".

When $n=1$, $9-4=5$ which is divisible by 5.

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $9^k-4^k=5N$ where N is an integer.

Then $9^{k+1}-4^{k+1}$

$=9\cdot9^k-4\cdot4^k$

$=9(5N+4^k)-4\cdot4^k$

$=9(5N)+5\cdot4^k$

$=5(9N+4^k)$

which is divisible by 5.

Hence $P(k+1)$ is true if $P(k)$ is true for any positive integer k. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

4. Let $P(n)$ be the proposition "23^n+10 is divisible by 11".

When $n=1$, $23^1+10=33=11(3)$ which is divisible by 11.

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $23^k+10=11N$ where N is an integer.

Then $23^{k+1}+10$

$=23(11N-10)+10$

$=23(11N)-220$

$=11(23N-20)$

which is divisible by 11.

Hence $P(k+1)$ is true if $P(k)$ is true for any positive integer k. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

5. Let $P(n)$ be the proposition "6^n-5n+4 divisible by 5".

When $n=1$, $6-5+4=5$ which is divisible by

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $6^k-5k+4=5N$ where N is an integer.

Then

$6^{k+1}-5(k+1)+4$

$=6\cdot6^k-5k-1$

$=6(5N+5k-4)-5k-1$

$=30N+25k-25$

$=5(6N+5k-5)$

which is divisible by 5.

Hence $P(k+1)$ is true if $P(k)$ is true for an positive integer k. By the principle mathematical induction, $P(n)$ is true for a positive integers n.

6. Let $P(n)$ be the proposition "$n(n+1)(n+2)$ divisible by 3".

When $n=1$, $(1+1)(1+2)=6$ which is divisib by 3.

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $k(k+1)(k+2)=3N$ where N is an integer.

Then

$(k+1)(k+2)(k+3)$

$=k(k+1)(k+2)+3(k+1)(k+2)$

$=3N+3(k+1)(k+2)$

$=3[N+(k+1)(k+2)]$

which is divisible by 3.

Hence $P(k+1)$ is true if $P(k)$ is true for an positive integer k. By the principle o mathematical induction, $P(n)$ is true for a positive integers n.

7. No solution is provided for the H.K.C.E.E question because of the copyright reasons.

8. Let $P(n)$ be the proposition "4^n+5^n is divisibl by 9".

When $n=1$, $4+5=9$ which is divisible by 9.

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive odd numbe k.

i.e. $4^k+5^k=9N$ where N is an integer.

For $n=k+2$

Then $4^{k+2}+5^{k+2}$

$=4^k\cdot 16+5^k\cdot 25$

$=16(9N-5^k)+25\cdot 5^k$

$=16(4^k+5^k)+9\cdot 5^k$

$=16(9N)+9\cdot 5^k$

$=9(16N+5^k)$

which is divisible by 9.

Hence $P(k+2)$ is true for any positive odd number k. By the principle of mathematical induction, $P(n)$ is true for all positive odd numbers n.

9. (a) Let $P(n)$ be the proposition "$x^n-nxy^{n-1}+(n-1)y^n$ is divisible by $(x-y)^2$".

When $n=2$, $x^2-2xy+y^2=(x-y)^2$ which is divisible by $(x-y)^2$.

$\therefore$ $P(2)$ is true.

Assume $P(k)$ is true for any positive integer $k\geq 2$.

i.e. $x^k-kxy^{k-1}+(k-1)y^k=(x-y)^2N$

where N is a polynomial.

Then

$x^{k+1}-(k+1)xy^k+ky^{k+1}$

$=x\cdot x^k-(k+1)xy^k+ky^{k+1}$

$=x[(x-y)^2N+kxy^{k-1}-(k-1)y^k]$
$\quad -(k+1)xy^k+ky^{k+1}$

$=x(x-y)^2N+kx^2y^{k-1}-(k-1)xy^k$
$\quad -kxy^k-xy^k+ky^{k+1}$

$=x(x-y)^2N+ky^{k-1}(x^2-2xy+y^2)$

$=x(x-y)^2N+(x-y)^2ky^{k-1}$

$=(x-y)^2(xN+ky^{k-1})$

which is divisible by $(x-y)^2$.

Hence $P(k+1)$ is true if $P(k)$ is true for any positive integer $k\geq 2$. By the principle of mathematical induction, $P(k)$ is true for all positive integers $n\geq 2$.

(b) (i) By **(a)**, put $x=9$, $y=4$

$\therefore$ $9^n-n(9)4^{n-1}+(n-1)4^n$ is divisible by $(9-4)^2$.

$4\cdot 9^n-9n\cdot 4^n+(n-1)4^{n+1}$ is divisible by 25.

$4\cdot 9^n-(9n-4n+4)\cdot 4^n$ is divisible by 25.

$\therefore$ $4\cdot 9-(5n+4)\cdot 4^n$ is divisible by 25.

(ii) By **(a)**, put $x=7$, $y=4$

$7^n-n(7)4^{n-1}+(n-1)4^n$

$=7^n-(7n-4n+4)4^{n-1}$

$=7^n-(3n+4)\cdot 4^{n-1}$

is divisible by $(7-4)^2$.

$\therefore$ $7^n-(3n+4)\cdot 4^{n-1}$ is divisible by 9.

10. Let $P(n)$ be the proposition "$a_n=\frac{1}{2}n(n-1)+1$".

When $n=1$, L.H.S. $=a_1=1$

R.H.S. $=\frac{1}{2}(1)(1-1)+1=1$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $a_k=\frac{1}{2}k(k-1)+1$

Then

$a_{k+1}=a_k+k$

$=\frac{1}{2}k(k-1)+1+k$

$=\frac{1}{2}k(k-1+2)+1$

$=\frac{1}{2}k(k+1)+1$

$=\frac{1}{2}(k+1)[(k+1)-1]+1$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

11. Let $P(n)$ be the proposition "$a_n=2n-1$".

When $n=1$,

L.H.S. $=a_1=1$

R.H.S. $=2(1)-1=1$

$\therefore$ $P(1)$ is true.

When $n=2$,

L.H.S. $=a_2=3$

R.H.S. $=2(2)-1=3$

$\therefore$ $P(2)$ is true.

Assume $P(k)$ and $P(k+1)$ are true for any positive integer k.

i.e. $a_k=2k-1$

$a_{k+1}=2(k+1)-1$

Then $a_{k+2}=2a_{k+1}-a_k$

$=2[2(k+1)-1]-(2k-1)$

$=4k+4-2-2k+1$

$=2k+3$

$=2(k+2)-1$

Thus assuming $P(k)$ and $P(k+1)$ are true for any positive integer k, $P(k+2)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

Revision Exercise 3 (p.76)

1. Let $P(n)$ be the proposition "$5+2\cdot 5^2+3\cdot 5^3+\cdots+n\cdot 5^n=\frac{(4n-1)5^{n+1}+5}{16}$".

When $n=1$, L.H.S. $=5$

R.H.S. $=\frac{(4-1)5^2+5}{16}=5$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $5+2\cdot 5^2+3\cdot 5^3+\cdots+k\cdot 5^k$

$=\frac{(4k-1)5^{k+1}+5}{16}$

Then $5+2\cdot 5^2+3\cdot 5^3+\cdots+k\cdot 5^k+(k+1)\cdot 5^{k+1}$

$=\frac{(4k-1)5^{k+1}+5}{16}+(k+1)\cdot 5^{k+1}$

$=\frac{(4k-1)5^{k+1}+5+16(k+1)5^{k+1}}{16}$

$=\frac{(4k-1+16k+16)5^{k+1}+5}{16}$

$=\frac{(20k+15)5^{k+1}+5}{16}$

$=\frac{(4k+3)5^{(k+1)+1}+5}{16}$

$=\frac{[4(k+1)-1]5^{(k+1)+1}+5}{16}$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

2. Let $P(n)$ be the proposition "$a+ar+ar^2+\cdots+ar^{n-1}=\frac{a(r^n-1)}{r-1}$".

When $n=1$, L.H.S. $=a$

R.H.S. $=\frac{a(r^1-1)}{r-1}=a$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $a+ar+ar^2+\cdots+ar^{k-1}=\frac{a(r^k-1)}{r-1}$

Then $a+ar+ar^2+\cdots+ar^{k-1}+ar^k$

$=\frac{a(r^k-1)}{r-1}+ar^k$

$=\frac{ar^k-a+ar^{k+1}-ar^k}{r-1}$

$=\frac{a(r^{k+1}-1)}{r-1}$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

3. Let $P(n)$ be the proposition "$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$".

When $n=1$, L.H.S. $=\frac{1}{1\cdot 2\cdot 3}=\frac{1}{6}$

R.H.S. $=\frac{(1+3)}{4(1+1)(1+2)}=\frac{1}{6}$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{k(k+1)(k+2)}$

$=\frac{k(k+3)}{4(k+1)(k+2)}$

Then $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{k(k+1)(k+2)}$
$+\frac{1}{(k+1)[(k+1)+1][(k+1)+2]}$

$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$

$=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$

$=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$

$=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$

$=\frac{(k+1)[(k+1)+3]}{4[(k+1)+1][(k+1)+2]}$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

4. Let $P(n)$ be the proposition
" $1^2-2^2+3^2-4^2+\cdots+(-1)^{n-1}n^2$
$=(-1)^{n-1}\frac{1}{2}n(n+1)$ ".

When $n=1$, L.H.S. $=1^2=1$

R.H.S. $=(-1)^0\frac{1}{2}(1+1)=1$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $1^2-2^2+\cdots+(-1)^{k-1}k^2=(-1)^{k-1}\frac{1}{2}k(k+1)$

Then

$$1^2-2^2+\cdots+(-1)^{k-1}k^2+(-1)^{(k+1)-1}(k+1)^2$$
$$=(-1)^{k-1}\frac{1}{2}k(k+1)+(-1)^k(k+1)^2$$
$$=(-1)^k\frac{1}{2}(k+1)(-k+2k+2)$$
$$=(-1)^k\frac{1}{2}(k+1)(k+2)$$
$$=(-1)^{(k+1)-1}\frac{1}{2}(k+1)[(k+1)+1]$$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

5. No solution is provided for the H.K.C.E.E. question because of the copyright reasons.

6. Let $P(n)$ be the proposition
" $T_1+T_2+\cdots+T_n=\frac{n(n+1)(n+5)}{3}$ ".

When $n=1$, L.H.S. $=T_1=1^2+3(1)=4$

R.H.S. $=\frac{(1+1)(1+5)}{3}=4$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $T_1+T_2+\cdots+T_k=\frac{k(k+1)(k+5)}{3}$

Then $T_1+T_2+\cdots+T_k+T_{k+1}$

$$=\frac{k(k+1)(k+5)}{3}+(k+1)^2+3(k+1)$$
$$=\frac{k+1}{3}[k(k+5)+3(k+1)+9]$$
$$=\frac{k+1}{3}[(k+6)(k+2)]$$
$$=\frac{(k+1)[(k+1)+1][(k+1)+5]}{3}$$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

7. No solution is provided for the H.K.C.E.E. question because of the copyright reasons.

8. Let $P(n)$ be the proposition " $8^n+2\cdot 7^n-1$ is divisible by 7".

When $n=1$, $8+2\cdot 7-1=21=7\cdot 3$ which is divisible by 7.

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $8^k+2\cdot 7^k-1=7N$ where N is an integer.

Then $8^{k+1}+2\cdot 7^{k+1}-1$

$$=8\cdot 8^k+14\cdot 7^k-1$$
$$=8(7N-2\cdot 7^k+1)+14\cdot 7^k-1$$
$$=7(8N)-2\cdot 7^k+7$$
$$=7(8N-2\cdot 7^{k-1}+1)$$

which is divisible by 7.

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

9. Let $P(n)$ be the proposition " $6^{n+2}+7^{2n+1}$ is divisible by 43".

When $n=1$, $6^3+7^3=559=43\times 13$ which is divisible by 43.

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $6^{k+2}+7^{2k+1}=43N$ where N is an integer.

Then $6^{k+3}+7^{2k+3}$

$$=6\cdot 6^{k+2}+49\cdot 7^{2k+1}$$
$$=6(43N-7^{2k+1})+49\cdot 7^{2k+1}$$
$$=6(43N)+43\cdot 7^{2k+1}$$
$$=43(6N+7^{2k+1})$$

which is divisible by 43.

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

10. (a) Let $P(n)$ be the proposition
" $1^3+2^3+3^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2$ ".

When $n=1$, L.H.S. $=1^3=1$

R.H.S. $=\frac{1}{4}(1+1)^2=1$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $1^3+2^3+\cdots+k^3=\frac{1}{4}k^2(k+1)^2$

Then $1^3+2^3+\cdots+k^3+(k+1)^3$

$$=\frac{1}{4}k^2(k+1)^2+(k+1)^3$$
$$=\frac{1}{4}(k+1)^2(k^2+4k+4)$$
$$=\frac{1}{4}(k+1)^2(k+2)^2$$
$$=\frac{1}{4}(k+1)^2[(k+1)+1]^2$$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

(b) $(1^3-1)+(2^3-2)+\cdots+(n^3-n)$

$$=1^3+2^3+\cdots+n^3-(1+2+\cdots+n)$$
$$=\frac{1}{4}n^2(n+1)^2-\frac{1}{2}n(n+1)$$
$$=\frac{1}{4}n(n+1)[n(n+1)-2]$$
$$=\frac{1}{4}n(n+1)(n^2+n-2)$$
$$=\frac{1}{4}(n-1)n(n+1)(n+2)$$

11. No solution is provided for the H.K.C.E.E. question because of the copyright reasons.

12. Let $P(n)$ be the proposition " a^n-b^n is divisible by $a-b$ ".

When $n=1$, $a^1-b^1=a-b$ which is divisible by $a-b$.

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $a^k-b^k=(a-b)N$ where N is a polynomial.

Then $a^{k+1}-b^{k+1}$

$$=a\cdot a^k-b\cdot b^k$$
$$=a[(a-b)N+b^k]-b\cdot b^k$$
$$=a(a-b)N+a\cdot b^k-b\cdot b^k$$
$$=a(a-b)N+(a-b)b^k$$
$$=(a-b)(aN+b^k)$$

which is divisible by $(a-b)$.

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

13. Let $P(n)$ be the proposition " $f(n)=3^{2n}-1$ is divisible by 8".

When $n=1$, $f(1)=3^2-1=8$ which is divisible by 8.

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $f(k)=3^{2k}-1=8N$ where N is an integer.

Then $f(k+1)=3^{2(k+1)}-1$

$$=3^{2k}\cdot 9-1$$
$$=(8N+1)9-1$$
$$=8N\cdot 9+8$$
$$=8(9N+1)$$

which is divisible by 8.

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

14 – 16. No solutions are provided for the H.K.C.E.E. questions because of the copyright reasons.

Enrichment 3 (p.78)

1. (a) $T_n=T_{n-1}+1$ for $n\geq 3$

Let $P(n)$ be the proposition " $T_n=n$ ".

When $n=2$,

L.H.S. $=T_2=2$

R.H.S. $=2$

$\therefore$ $P(2)$ is true.

Assume $P(k)$ is true for any positive integer $k\geq 2$.

i.e. $T_k=k$

Then $T_{k+1}=T_k+1=k+1$

Thus assuming $P(k)$ is true for any positive integer $k\geq 2$, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers $n\geq 2$.

(b) $Q_n=Q_{n-1}+(n-1)$ for $n\geq 3$

Let $P(n)$ be the proposition " $Q_n=\frac{n(n-1)}{2}$ ".

When $n=2$,

L.H.S. $=Q_2=1$

R.H.S. $=\frac{2(2-1)}{2}=1$

$\therefore$ $P(2)$ is true.

Assume $P(k)$ is true for any positive integer $k \geq 2$.

i.e. $Q_k = \dfrac{k(k-1)}{2}$

Then $Q_{k+1} = Q_k + [(k+1-1)]$

$= \dfrac{k(k-1)}{2} + k$

$= \dfrac{k(k-1)+2k}{2}$

$= \dfrac{k(k-1+2)}{2}$

$= \dfrac{k(k+1)}{2}$

Thus assuming $P(k)$ is true for any positive integer $k \geq 2$, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers $n \geq 2$.

2. (a) $x^2 - x - 1 = 0$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$\because \quad \alpha > \beta$

$\therefore \quad \alpha = \dfrac{1+\sqrt{5}}{2}, \ \beta = \dfrac{1-\sqrt{5}}{2}$

(b) Let $P(n)$ be the proposition "$a_n = \dfrac{1}{\sqrt{5}}(\alpha^n - \beta^n)$".

When $n = 1$,

L.H.S. $= a_1 = 1$

R.H.S. $= \dfrac{1}{\sqrt{5}}(\alpha - \beta)$

$= \dfrac{1}{\sqrt{5}}[(\dfrac{1+\sqrt{5}}{2}) - (\dfrac{1-\sqrt{5}}{2})]$

$= 1$

$\therefore \quad P(1)$ is true.

When $n = 2$,

L.H.S. $= a_2 = 1$

R.H.S.

$= \dfrac{1}{\sqrt{5}}(\alpha^2 - \beta^2)$

$= \dfrac{1}{\sqrt{5}}[(\dfrac{1+\sqrt{5}}{2})^2 - (\dfrac{1-\sqrt{5}}{2})^2]$

$= \dfrac{1}{\sqrt{5}}(\dfrac{1+\sqrt{5}}{2} + \dfrac{1-\sqrt{5}}{2})(\dfrac{1+\sqrt{5}}{2} - \dfrac{1-\sqrt{5}}{2})]$

$= 1$

$\therefore \quad P(2)$ is true.

Assume $P(k)$ and $P(k+1)$ are true for any positive integer $k \geq 2$.

i.e. $a_k = \dfrac{1}{\sqrt{5}}(\alpha^k - \beta^k)$

$a_{k+1} = \dfrac{1}{\sqrt{5}}(\alpha^{k+1} - \beta^{k+1})$

Then a_{k+2}

$= a_{k+1} + a_k$

$= \dfrac{1}{\sqrt{5}}(\alpha^{k+1} - \beta^{k+1}) + \dfrac{1}{\sqrt{5}}(\alpha^k - \beta^k)$

$= \dfrac{1}{\sqrt{5}}(\alpha^{k+1} - \beta^{k+1} + \alpha^k - \beta^k)$

$= \dfrac{1}{\sqrt{5}}[\alpha^k(\alpha+1) - \beta^k(\beta+1)]$

$= \dfrac{1}{\sqrt{5}}[\alpha^k(\dfrac{1+\sqrt{5}}{2}+1) - \beta^k(\dfrac{1-\sqrt{5}}{2}+1)]$

$= \dfrac{1}{\sqrt{5}}[\alpha^k(\dfrac{3+\sqrt{5}}{2}) - \beta^k(\dfrac{3-\sqrt{5}}{2})]$

$= \dfrac{1}{\sqrt{5}}[\alpha^k(\dfrac{1+\sqrt{5}}{2})^2 - \beta^k(\dfrac{1-\sqrt{5}}{2})^2]$

$= \dfrac{1}{\sqrt{5}}(\alpha^k \cdot \alpha^2 - \beta^k \cdot \alpha^2)$

$= \dfrac{1}{\sqrt{5}}(\alpha^{k+2} - \beta^{k+2})$

Thus assuming $P(k)$ and $P(k+1)$ are true for any positive integer k, $P(k+2)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

3. No solution is provided for the H.K.C.E.E. question because of the copyright reasons.

Classwork 1 (p.68)

Let $P(n)$ be the proposition "$1+3+5+\cdots+(2n-1) = n^2$".

When $n = 1$,

L.H.S. $= 1$

R.H.S. $= 1^2 = 1$

$\therefore \quad P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $1+3+5+\cdots+(2k-1) = k^2$

Then $1+3+5+\cdots+(2k-1)+[2(k+1)-1]$

$= k^2 + (2k+1)$

$= (k+1)^2$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

Classwork 2 (p.71)

(a) Let $P(n)$ be the proposition "$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$".

When $n = 1$, L.H.S. $= 1^3 = 1$

R.H.S. $= 1^2[2(1)^2 - 1] = 1$

$\therefore \quad P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2-1)$

Then $1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 + [2(k+1)-1]^3$

$= k^2(2k^2-1) + (2k+1)^3$

$= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1$

$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$

$= (k+1)^2(2k^2 + 4k + 1)$

$= (k+1)^2[(2k^2 + 4k + 2) - 1]$

$= (k+1)^2[2(k^2 + 2k + 1) - 1]$

$= (k+1)^2[2(k+1)^2 - 1]$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

(b) $2^3 + 6^3 + 10^3 + \cdots + 38^3$

$= (2 \cdot 1)^3 + (2 \cdot 3)^3 + (2 \cdot 5)^3 + \cdots + (2 \cdot 19)^3$

$= 2^3 \cdot 1^3 + 2^3 \cdot 3^3 + 2^3 \cdot 5^3 + \cdots + 2^3 \cdot 19^3$

$= 8(1^3 + 3^3 + 5^3 + \cdots + 19^3)$

$= 8[10^2(2 \cdot 10^2 - 1)]$

$= 8[100(200-1)]$

$= \underline{\underline{159\ 200}}$

Classwork 3 (p.74)

1. Let $P(n)$ be the proposition "$7^n + 3n + 8$ is divisible by 9".

When $n = 1$, $7^1 + 3(1) + 8 = 18 = 9 \cdot 2$ which is divisible by 9.

$\therefore \quad P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $7^k + 3k + 8 = 9N$, where N is an integer.

Then $7^{k+1} + 3(k+1) + 8$

$= 7^k \cdot 7 + 3k + 3 + 8$

$= (9N - 3k - 8)7 + 3k + 3 + 8$

$= 63N - 21k - 56 + 3k + 11$

$= 63N - 18k - 45$

$= 9(7N - 2k - 5)$

which is divisible by 9.

Hence $P(k+1)$ is true if $P(k)$ is true for a positive integer k. By the principle mathematical induction, $P(n)$ is true for positive integers n.

2. Let $P(n)$ be the proposition "$a^{2n-1} + b^{2n-1}$ divisible by $a+b$".

When $n = 1$, $a^1 + b^1 = a + b$ which is divisible $a+b$.

$\therefore \quad P(1)$ is true.

Assume $P(k)$ is true for any positive integer k

i.e. $a^{2k-1} + b^{2k-1} = (a+b)N$, where N is polynomial.

Then $a^{2(k+1)-1} + b^{2(k+1)-1}$

$= a^{2k+1} + b^{2k+1}$

$= a^{2k-1+2} + b^{2k-1+2}$

$= a^{2k-1} \cdot a^2 + b^{2k-1} \cdot b^2$

$= [(a+b)N - b^{2k-1}]a^2 + b^{2k-1} \cdot b^2$

$= a^2(a+b)N - a^2b^{2k-1} + b^2b^{2k-1}$

$= a^2(a+b)N + b^{2k-1}(b^2 - a^2)$

$= a^2(a+b)N + b^{2k-1}(b-a)(b+a)$

$= (a+b)[a^2N + (b-a)b^{2k-1}]$

which is divisible by $a+b$.

Hence $P(k+1)$ is true if $P(k)$ is true for an positive integer k. By the principle o mathematical induction, $P(n)$ is true for al positive integers n.