CHAPTER 6

Exercise 6A (p.147)

1. (a)
$$\begin{aligned}\sin 105^\circ &= \sin(60^\circ+45^\circ)\\ &= \sin 60^\circ\cos 45^\circ+\cos 60^\circ\sin 45^\circ\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \underline{\underline{\frac{\sqrt{6}+\sqrt{2}}{4}}}\end{aligned}$$

(b)
$$\begin{aligned}\tan 165^\circ &= \tan(180^\circ-15^\circ)\\ &= -\tan 15^\circ\\ &= -\tan(45^\circ-30^\circ)\\ &= -\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}\\ &= -\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=-\frac{\sqrt{3}-1}{\sqrt{3}+1}\\ &= \underline{\underline{\sqrt{3}-2}}\end{aligned}$$

(c)
$$\begin{aligned}\cos 285^\circ &= \cos(360^\circ-75^\circ)\\ &= \cos 75^\circ\\ &= \cos(45^\circ+30^\circ)\\ &= \cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \underline{\underline{\frac{\sqrt{6}-\sqrt{2}}{4}}}\end{aligned}$$

2. (a)
$$\begin{aligned}\sin 3A\cos A-\cos 3A\sin A &= \sin(3A-A)\\ &= \underline{\underline{\sin 2A}}\end{aligned}$$

(b)
$$\begin{aligned}&\sin 40^\circ\cos 20^\circ+\cos 40^\circ\sin 20^\circ\\ &= \sin(40^\circ+20^\circ)\\ &= \sin 60^\circ\\ &= \underline{\underline{\frac{\sqrt{3}}{2}}}\end{aligned}$$

(c)
$$\begin{aligned}\cos 2A\cos A+\sin 2A\sin A &= \cos(2A-A)\\ &= \underline{\underline{\cos A}}\end{aligned}$$

(d) $\cos B\cos 45^\circ-\sin B\sin 45^\circ=\underline{\underline{\cos(B+45^\circ)}}$

(e)
$$\begin{aligned}&\cos(A+B)+\cos(A-B)\\ &= \cos A\cos B-\sin A\sin B\\ &\quad +\cos A\cos B+\sin A\sin B\\ &= \underline{\underline{2\cos A\cos B}}\end{aligned}$$

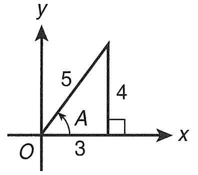
(f)
$$\begin{aligned}&\sin(\alpha+\beta)\cos\beta-\cos(\alpha+\beta)\sin\beta\\ &= \sin[(\alpha+\beta)-\beta]\\ &= \underline{\underline{\sin\alpha}}\end{aligned}$$

3. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=1$

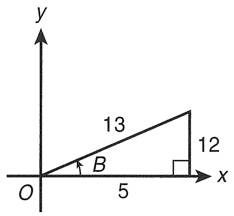
$A+B=45^\circ$ or 225° (rejected)

$\sin(A+B)=\underline{\underline{\dfrac{\sqrt{2}}{2}}}$

4.



$\cos A=\dfrac{3}{5},\ \sin A=\dfrac{4}{5}$



$\cos B=\dfrac{5}{13},\ \sin B=\dfrac{12}{13}$

$$\begin{aligned}\sin(A+B) &= \sin A\cos B+\cos A\sin B\\ &= \frac{4}{5}\cdot\frac{5}{13}+\frac{3}{5}\cdot\frac{12}{13}\\ &= \underline{\underline{\frac{56}{65}}}\end{aligned}$$

$$\begin{aligned}\cos(A-B) &= \cos A\cos B+\sin A\sin B\\ &= \frac{3}{5}\cdot\frac{5}{13}+\frac{4}{5}\cdot\frac{12}{13}\\ &= \underline{\underline{\frac{63}{65}}}\end{aligned}$$

5. $\cot A=2,\ \cot B=2,\ \cot C=8$

$\tan A=\dfrac{1}{2},\ \tan B=\dfrac{1}{5},\ \tan C=\dfrac{1}{8}$

$\tan(B+C)=\dfrac{\tan B+\tan C}{1-\tan B\tan C}=\dfrac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\cdot\frac{1}{8}}=\dfrac{1}{3}$

$\tan(B+C)>0,\ \therefore\ B+C$ is still acute

$\tan(A+B+C)$

$=\dfrac{\tan A+\tan(B+C)}{1-\tan A\tan(B+C)}=\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=1$

$\therefore\quad A+B+C=\underline{\underline{45^\circ}}$

6.
$$\begin{aligned}\tan A+\tan B &= \frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}\\ &= \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B}\\ &= \frac{\sin(A+B)}{\cos A\cos B}\end{aligned}$$

7.
$$\begin{aligned}&\sin(x+y)\sin(x-y)\\ &= (\sin x\cos y+\cos x\sin y)(\sin x\cos y-\cos x\sin y)\\ &= (\sin x\cos y)^2-(\cos x\sin y)^2\\ &= \sin^2 x(1-\sin^2 y)-(1-\sin^2 x)\sin^2 y\\ &= \sin^2 x-\sin^2 y\end{aligned}$$

8.
$$\begin{aligned}\sin x\cos(x-y)-\cos x\sin(x-y) &= \sin[x-(x-y)]\\ &= \sin y\end{aligned}$$

9.
$$\begin{aligned}&\frac{\sin B+\sin A\cos(A+B)}{\cos B-\sin A\sin(A+B)}\\ &= \frac{\sin[(A+B)-A]+\sin A\cos(A+B)}{\cos[(A+B)-A]-\sin A\sin(A+B)}\\ &= [\sin(A+B)\cos A-\cos(A+B)\sin A\\ &\quad +\sin A\cos(A+B)]\div[\cos(A+B)\cos A\\ &\quad +\sin(A+B)\sin A-\sin A\sin(A+B)]\\ &= \frac{\sin(A+B)\cos A}{\cos(A+B)\cos A}\\ &= \tan(A+B)\end{aligned}$$

10.
$$\begin{aligned}\cot(A+B) &= \frac{1}{\tan(A+B)}\\ &= \frac{1-\tan A\tan B}{\tan A+\tan B}\\ &= \frac{1-\frac{1}{\cot A\cot B}}{\frac{1}{\cot A}+\frac{1}{\cot B}}\\ &= \frac{\frac{\cot A\cot B-1}{\cot A\cos B}}{\frac{\cot A+\cos B}{\cot A\cot B}}\\ &= \frac{\cot A\cot B-1}{\cot A+\cot B}\end{aligned}$$

11.
$$\begin{aligned}\cos 3x\cos x+\sin 3x\sin x &= \frac{1}{2}\\ \cos(3x-x) &= \frac{1}{2}\\ \cos 2x &= \frac{1}{2}\end{aligned}$$

$2x=\dfrac{\pi}{3},\dfrac{5\pi}{3},\dfrac{7\pi}{3},\dfrac{11\pi}{3}$

$x=\underline{\underline{\dfrac{\pi}{6},\dfrac{5\pi}{6},\dfrac{7\pi}{6},\dfrac{11\pi}{6}}}$

12.
$$\begin{aligned}(\sin 2x-1)\cos x &= \cos 2x\sin x\\ \sin 2x\cos x-\cos 2x\sin x &= \cos x\\ \sin(2x-x) &= \cos x\\ \sin x &= \cos x\\ \tan x &= 1\\ x &= \underline{\underline{\frac{\pi}{4},\frac{5\pi}{4}}}\end{aligned}$$

13.
$$\begin{aligned}\tan x+\tan 3x &= \sqrt{3}-\sqrt{3}\tan x\tan 3x\\ \tan x+\tan 3x &= \sqrt{3}(1-\tan x\tan 3x)\\ \frac{\tan x+\tan 3x}{1-\tan x\tan 3x} &= \sqrt{3}\\ \tan(x+3x) &= \sqrt{3}\\ \tan 4x &= \sqrt{3}\end{aligned}$$

$4x=\dfrac{\pi}{3},\dfrac{4\pi}{3},\dfrac{7\pi}{3},\dfrac{10\pi}{3},\dfrac{13\pi}{3},\dfrac{16\pi}{3},\dfrac{19\pi}{3},\dfrac{22\pi}{3}$

$x=\dfrac{\pi}{12},\dfrac{\pi}{3},\dfrac{7\pi}{12},\dfrac{5\pi}{6},\dfrac{13\pi}{12},\dfrac{4\pi}{3},\dfrac{19\pi}{12},\dfrac{11\pi}{6}$

$\because\quad \tan 3(\dfrac{5\pi}{6})$ and $\tan 3(\dfrac{11\pi}{6})$ are undefined.

$\therefore\quad x=\underline{\underline{\dfrac{\pi}{12},\dfrac{\pi}{3},\dfrac{7\pi}{12},\dfrac{13\pi}{12},\dfrac{4\pi}{3},\dfrac{19\pi}{12}}}$

14.
$$\begin{aligned}&(\sin x+\cos x)^2=2\sin(\frac{\pi}{4}+x)\sin(\frac{\pi}{4}-x)\\ &\sin^2 x+2\sin x\cos x+\cos^2 x\\ &= 2(\sin\frac{\pi}{4}\cos x+\cos\frac{\pi}{4}\sin x)\\ &\quad(\sin\frac{\pi}{4}\cos x-\cos\frac{\pi}{4}\sin x)\\ &= 2(\sin^2\frac{\pi}{4}\cos^2 x-\cos^2\frac{\pi}{4}\sin^2 x)\\ &= 2(\frac{1}{2}\cos^2 x-\frac{1}{2}\sin^2 x)\\ &= \cos^2 x-\sin^2 x\end{aligned}$$

$\therefore\quad 2\sin^2 x+2\sin x\cos x=0$

$\qquad \sin x(\sin x+\cos x)=0$

$\therefore\quad \sin x=0\quad$ or $\quad\tan x=-1$

$\qquad x=0,\ \pi\quad$ or $\quad x=\dfrac{3}{4}\pi,\ \dfrac{7}{4}\pi$

$\therefore\quad x=\underline{\underline{0,\dfrac{3\pi}{4},\pi,\dfrac{7\pi}{4}}}$

15. $AD=3DE,\ BD=2DE$

$\tan a=\dfrac{DE}{AD}=\dfrac{1}{3},\ \tan b=\dfrac{DE}{BD}=\dfrac{1}{2}$

$\tan(a+b)=\dfrac{\tan a+\tan b}{1-\tan a\tan b}=\dfrac{\frac{1}{3}+\frac{1}{2}}{1-(\frac{1}{3})(\frac{1}{2})}=1;$

$\because\quad a$ and b are both acute.

$\therefore\quad a+b=45^\circ$

$\tan c=\dfrac{DE}{CD}=1,\ c=45^\circ$

$\therefore\quad a+b+c=90^\circ$

16. (a)
$$\begin{aligned}\sin A &= \sin[180^\circ-(B+C)]\\ &= \sin(B+C)\\ &= \sin B\cos C+\cos B\sin C\end{aligned}$$

(b)
$$\begin{aligned}\cos A &= \cos[180^\circ-(B+C)]\\ &= -\cos(B+C)\\ &= -(\cos B\cos C-\sin B\sin C)\end{aligned}$$

17. (a) $\sin(\alpha+\beta)=k\sin(\alpha-\beta)$

$\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$=k(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$
$\cos\alpha\sin\beta+k\cos\alpha\sin\beta$
$=k\sin\alpha\cos\beta-\sin\alpha\cos\beta$
$(k+1)\cos\alpha\sin\beta=(k-1)\sin\alpha\cos\beta$

$$\frac{k+1}{k-1}=\frac{\sin\alpha\cos\beta}{\cos\alpha\sin\beta}=\frac{\tan\alpha}{\tan\beta}$$

$$\therefore\quad \tan\alpha=\frac{k+1}{k-1}\tan\beta$$

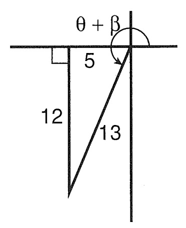
(b) $\sin(\theta+\frac{\pi}{3})=3\sin(\frac{2\pi}{3}+\theta)=3\sin(\frac{\pi}{3}-\theta)$

$\therefore\quad \tan\frac{\pi}{3}=\frac{3+1}{3-1}\tan\theta$ (by **(a)**)

$$\tan\theta=\frac{2}{4}\tan\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$

$\therefore\quad \theta=\underline{\underline{0.713\ 7}}$ or $\underline{\underline{3.855\ 3}}$ (corr. to 4 d.p.)

(c) Since α, β are obtuse and $\tan(\alpha+\beta)>0$, $(\alpha+\beta)$ must lie on quadrant III.



Therefore, $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ are both negative.

$$\therefore\quad \sin(\alpha+\beta)=-\frac{12}{13}$$
$$\cos(\alpha+\beta)=-\frac{5}{13}$$

One required quadratic equation is

$$(x+\frac{12}{13})(x+\frac{5}{13})=0$$
$$(13x+12)(13x+5)=0$$
$$\underline{\underline{169x^2+221x+60=0}}$$

18. (a) Sum of the roots $=\tan A+\tan B=\underline{\underline{4}}$

Product of the roots $=\tan A\tan B=\underline{\underline{2}}$

(b) $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{4}{1-2}=\underline{\underline{-4}}$

(c) $\cot A+\cot B$

$$=\frac{1}{\tan A}+\frac{1}{\tan B}=\frac{\tan A+\tan B}{\tan A\tan B}=\frac{4}{2}=\underline{\underline{2}}$$

19. $\tan\alpha\tan\beta=\frac{1}{4}$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=2$$

$\therefore\quad \tan\alpha+\tan\beta=2(1-\frac{1}{4})=\frac{3}{2}$

$\therefore$ They are the roots of the equation

$$x^2-\frac{3}{2}x+\frac{1}{4}=0$$

i.e. $4x^2-6x+1=0$

20. (a) Sum of the roots $=\tan\alpha+\tan\beta=-12$

Product of the roots $=\tan\alpha\tan\beta=6$

Since their sum is negative and their product is positive, they must be both negative. Therefore α, β are both obtuse.

(b) $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$=\frac{-12}{1-6}=\underline{\underline{\frac{12}{5}}}$$

21. Sum of the roots $=\tan\alpha+\tan\beta=3$

Product of the roots $=\tan\alpha\tan\beta=-3$

$\therefore\quad \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{3}{4}$

$\therefore\quad \sin^2(\alpha+\beta)-3\sin(\alpha+\beta)\cos(\alpha+\beta)$
$-3\cos^2(\alpha+\beta)$

$$=\cos^2(\alpha+\beta)[\frac{\sin^2(\alpha+\beta)}{\cos^2(\alpha+\beta)}-3\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}-3]$$
$$=\frac{\tan^2(\alpha+\beta)-3\tan(\alpha+\beta)-3}{1+\tan^2(\alpha+\beta)}$$
$$=\frac{1}{1+(\frac{3}{4})^2}[(\frac{3}{4})^2-3(\frac{3}{4})-3]$$
$$=\underline{\underline{-3}}$$

22. (a) $A+B+C=180°$

$\therefore\quad \frac{A}{2}=90°-(\frac{B}{2}+\frac{C}{2})$

$\therefore\quad \cot\frac{A}{2}=\cot[90°-(\frac{B}{2}+\frac{C}{2})]=\tan(\frac{B}{2}+\frac{C}{2})$

(b) (i) $\cot\frac{A}{2}=\tan(\frac{B}{2}+\frac{C}{2})$ (by **(a)**)

$$=\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{1-\tan\frac{B}{2}\tan\frac{C}{2}}$$

(ii)
$$\cot\frac{A}{2}=\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{1-\tan\frac{B}{2}\tan\frac{C}{2}}=\frac{\frac{1}{\cot\frac{B}{2}}+\frac{1}{\cot\frac{C}{2}}}{1-\frac{1}{\cot\frac{B}{2}}\frac{1}{\cot\frac{C}{2}}}=\frac{\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot\frac{B}{2}\cot\frac{C}{2}-1}$$

$$\cot\frac{A}{2}(\cot\frac{B}{2}\cot\frac{C}{2}-1)=\cot\frac{B}{2}+\cot\frac{C}{2}$$

$$\therefore\quad \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$$

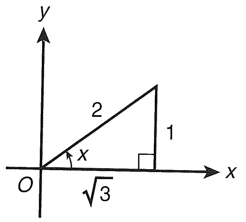
23.
$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$=\frac{\frac{\sin\alpha}{a-\cos\alpha}+\frac{a\sin\alpha}{1-a\cos\alpha}}{1-\frac{\sin\alpha}{a-\cos\alpha}\frac{a\sin\alpha}{1-a\cos\alpha}}$$
$$=\frac{\sin\alpha(1-a\cos\alpha)+a\sin\alpha(a-\cos\alpha)}{(a-\cos\alpha)(1-a\cos\alpha)-a\sin^2\alpha}$$
$$=\frac{\sin\alpha-a\sin\alpha\cos\alpha+a^2\sin\alpha-a\sin\alpha\cos\alpha}{a-\cos\alpha-a^2\cos\alpha+a\cos^2\alpha-a\sin^2\alpha}$$
$$=\frac{\sin\alpha-2a\sin\alpha\cos\alpha+a^2\sin\alpha}{-\cos\alpha-a^2\cos\alpha+a\cos^2\alpha+a(1-\sin^2\alpha)}$$
$$=\frac{\sin\alpha(1-2a\cos\alpha+a^2)}{-\cos\alpha(1-2a\cos\alpha+a^2)}$$
$$=-\tan\alpha$$

Exercise 6B (p.152)

1. $\cos 2x=1-2\sin^2 x=1-2(\frac{3}{5})^2=\underline{\underline{\frac{7}{25}}}$

2. $\cos x=\frac{\sqrt{3}}{2}$

$\sin x=\frac{1}{2}$

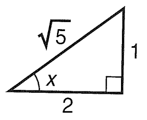


$\sin 2x=2\sin x\cos x$

$$=2(\frac{1}{2})(\frac{\sqrt{3}}{2})=\underline{\underline{\frac{\sqrt{3}}{2}}}$$

3. $\therefore\quad \tan x=\frac{1}{2}$, then $\sin x=\pm\frac{1}{\sqrt{5}}$,

$\cos x=\pm\frac{2}{\sqrt{5}}$



(a) when $\tan x>0$

$\sin x$ and $\cos x$ have the same sign.

$$\sin 2x=2\sin x\cos x=2(\pm\frac{1}{\sqrt{5}})(\pm\frac{2}{\sqrt{5}})=\underline{\underline{\frac{4}{5}}}$$

(b) $\cos 2x=\cos^2 x-\sin^2 x$

$$=(\pm\frac{2}{\sqrt{5}})^2-(\pm\frac{1}{\sqrt{5}})^2=\underline{\underline{\frac{3}{5}}}$$

(c) $\sin 4x=2\sin 2x\cos 2x$

$$=2(\frac{4}{5})(\frac{3}{5})\ \text{(by (a), (b))}=\underline{\underline{\frac{24}{25}}}$$

(d) $\cos 6x$

$$=\cos(4x+2x)$$
$$=\cos 4x\cos 2x-\sin 4x\sin 2x$$
$$=(\cos^2 2x-\sin^2 2x)\cos 2x-\sin 4x\sin 2x$$
$$=[(\frac{3}{5})^2-(\frac{4}{5})^2](\frac{3}{5})-(\frac{24}{25})(\frac{4}{5})$$
$$=\underline{\underline{-\frac{117}{125}}}$$

4.
$$\cos 4\theta=\frac{1}{4}$$
$$2\cos^2 2\theta-1=\frac{1}{4}$$
$$\cos^2 2\theta=\frac{5}{8}$$
$$\sin^2 2\theta=1-\frac{5}{8}=\frac{3}{8}$$

(a) $\cos^4\theta-\sin^4\theta$

$$=(\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)$$
$$=\cos 2\theta$$
$$=\pm\sqrt{\frac{5}{8}}$$
$$=\underline{\underline{\pm\frac{\sqrt{10}}{4}}}$$

(b) $\cos^4+\sin^4\theta$

$$=(\cos^2\theta+\sin^2\theta)^2-2\cos^2\theta\sin^2\theta$$
$$=1-\frac{1}{2}\sin^2 2\theta$$
$$=1-\frac{1}{2}(\frac{3}{8})$$
$$=\underline{\underline{\frac{13}{16}}}$$

5. (a)
$$\begin{aligned}&\frac{\sin 3\theta}{\sin\theta}-\frac{\cos 3\theta}{\cos\theta}\\&=\frac{\sin 3\theta\cos\theta-\cos 3\theta\sin\theta}{\sin\theta\cos\theta}\\&=\frac{\sin(3\theta-\theta)}{\frac{1}{2}\sin 2\theta}\\&=\frac{\sin 2\theta}{\frac{1}{2}\sin 2\theta}\\&=\underline{\underline{2}}\end{aligned}$$

(b)
$$\begin{aligned}&\frac{\cos^3\theta-\cos 3\theta}{\cos\theta}+\frac{\sin^3\theta+\sin 3\theta}{\sin\theta}\\&=\cos^2\theta-\frac{\cos 3\theta}{\cos\theta}+\sin^2\theta+\frac{\sin 3\theta}{\sin\theta}\\&=1+\frac{\sin 3\theta}{\sin\theta}-\frac{\cos 3\theta}{\cos\theta}\\&=1+2\ (\text{by }\mathbf{(a)})\\&=\underline{\underline{3}}\end{aligned}$$

6.
$$\begin{aligned}\frac{1-\cos 2x}{1+\cos 2x}&=\frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)}\\&=\frac{2\sin^2 x}{2\cos^2 x}\\&=\tan^2 x\end{aligned}$$

7.
$$\begin{aligned}&\sin^4 x+\cos^4\\&=(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x\\&=1-\frac{1}{2}\sin^2 2x\end{aligned}$$

8.
$$\begin{aligned}\frac{1}{1-\tan x}-\frac{1}{1+\tan x}&=\frac{(1+\tan x)-(1-\tan x)}{(1-\tan x)(1+\tan x)}\\&=\frac{2\tan x}{1-\tan^2 x}\\&=\tan 2x\end{aligned}$$

9.
$$\begin{aligned}2\sin x+\sin 2x&=2\sin x+2\sin x\cos x\\&=2\sin x(1+\cos x)\\&=2\sin x\frac{(1-\cos^2 x)}{1-\cos x}\\&=\frac{2\sin^3 x}{1-\cos x}\end{aligned}$$

10.
$$\begin{aligned}\frac{\sin x+\sin 2x}{1+\cos x+\cos 2x}&=\frac{\sin x+2\sin x\cos x}{1+\cos x+(2\cos^2 x-1)}\\&=\frac{\sin x(1+2\cos x)}{\cos x(1+2\cos x)}\\&=\tan x\end{aligned}$$

11.
$$\begin{aligned}&\tan(45^\circ+x)-\tan(45^\circ-x)\\&=\frac{1+\tan x}{1-\tan x}-\frac{1-\tan x}{1+\tan x}\\&=\frac{(1+\tan x)^2-(1-\tan x)^2}{(1+\tan x)(1-\tan x)}\\&=\frac{(1+\tan x-1+\tan x)(1+\tan x+1-\tan x)}{1-\tan^2 x}\\&=\frac{2(2\tan x)}{1-\tan^2 x}\\&=2\tan 2x\end{aligned}$$

12.
$$\begin{aligned}&3-4\cos 2\theta+\cos 4\theta\\&=3-4\cos 2\theta+(2\cos^2 2\theta-1)\\&=2\cos^2 2\theta-4\cos 2\theta+2\\&=2(\cos 2\theta-1)^2\\&=2[(1-2\sin^2\theta)-1)]^2\\&=2(-2\sin^2\theta)^2\\&=8\sin^4\theta\end{aligned}$$

13.
$$\begin{aligned}\tan\theta-\cot\theta&=\tan\theta-\frac{1}{\tan\theta}\\&=\frac{\tan^2\theta-1}{\tan\theta}\\&=-2\cdot\frac{1-\tan^2\theta}{2\tan\theta}\\&=-2\cdot\frac{1}{\tan 2\theta}\\&=-2\cot 2\theta\end{aligned}$$

14.
$$\begin{aligned}&\frac{1+\sin 2x-\cos 2x}{1+\sin 2x+\cos 2x}\\&=\frac{1+2\sin x\cos x-(1-2\sin^2 x)}{1+2\sin x\cos x+(2\cos^2 x-1)}\\&=\frac{2\sin x(\cos x+\sin x)}{2\cos x(\sin x+\cos x)}\\&=\tan x\end{aligned}$$

15.
$$\begin{aligned}3\cos 2x+5\sin x&=4\\3(1-2\sin^2 x)+5\sin x&=4\\6\sin^2 x-5\sin x+1&=0\\(2\sin x-1)(3\sin x-1)&=0\end{aligned}$$
$\sin x=\dfrac{1}{2}$ or $\sin x=\dfrac{1}{3}$

$x=30^\circ,\ 150^\circ$ or $x=19.5^\circ,\ 160.5^\circ$ (corr. to 1 d.p.)

$\therefore\ x=\underline{\underline{19.5^\circ,\ 30^\circ,\ 150^\circ,\ 160.5^\circ}}$

16.
$$\begin{aligned}3(1-\sin x)&=1+\cos 2x\\3-3\sin x&=1+(1-2\sin^2 x)\\2\sin^2 x-3\sin x+1&=0\\(2\sin x-1)(\sin x-1)&=0\end{aligned}$$
$\sin x=\dfrac{1}{2}$ or $\sin x=1$

$x=30^\circ,\ 150^\circ$ or $x=90^\circ$

$\therefore\ x=\underline{\underline{30^\circ,\ 90^\circ,\ 150^\circ}}$

17.
$$\begin{aligned}2\sin^2 x+\sin^2 2x&=2\\2\sin^2 x+(2\sin x\cos x)^2&=2\\\sin^2 x+2\sin^2 x(1-\sin^2 x)&=1\\2\sin^4 x-3\sin^2 x+1&=0\\(2\sin^2 x-1)(\sin^2 x-1)&=0\end{aligned}$$
$\sin^2 x=\dfrac{1}{2}$ or $\sin^2 x=1$

$\sin x=\pm\dfrac{1}{\sqrt{2}}$ or $\sin x=\pm 1$

$x=45^\circ,\ 135^\circ,\ 225^\circ,\ 315^\circ$ or $x=90^\circ,\ 270^\circ$

$\therefore\ x=\underline{\underline{45^\circ,\ 90^\circ,\ 135^\circ,\ 225^\circ,\ 270^\circ,\ 315^\circ}}$

18.
$$\begin{aligned}2\sin^2 x-\sin 2x&=2\\2(1-\sin^2 x)+\sin 2x&=0\\2\cos^2 x+\sin 2x&=0\\2\cos^2 x+2\sin x\cos x&=0\\\cos x(\cos x+\sin x)&=0\end{aligned}$$
$\cos x=0$ or $\tan x=-1$

$x=90^\circ,\ 270^\circ$ or $x=135^\circ,\ 315^\circ$

$\therefore\ x=\underline{\underline{90^\circ,\ 135^\circ,\ 270^\circ,\ 315^\circ}}$

19.
$$\begin{aligned}\sin x+\cos x+\sin 2x+\cos 2x+1&=0\\\sin x+\cos x+2\sin x\cos x+2\cos^2 x&=0\\\sin x(1+2\cos x)+\cos x(1+2\cos x)&=0\\(\sin x+\cos x)(1+2\cos x)&=0\end{aligned}$$
$\sin x+\cos x=0$ or $1+2\cos x=0$

$\tan x=-1$ or $\cos x=-\dfrac{1}{2}$

$x=135^\circ,\ 315^\circ$ or $x=120^\circ,\ 240^\circ$

$\therefore\ x=\underline{\underline{120^\circ,\ 135^\circ,\ 240^\circ,\ 315^\circ}}$

20.
$$\begin{aligned}&(1-\tan x)(1+\sin 2x)=1+\tan x\\&(1-\frac{\sin x}{\cos x})(1+2\sin x\cos x)=1+\frac{\sin x}{\cos x}\\&(\frac{\cos x-\sin x}{\cos x})(1+2\sin x\cos x)=\frac{\cos x+\sin x}{\cos x}\\&(\cos x-\sin x)(1+2\sin x\cos x)=\cos x+\sin x\\&(\cos x-\sin x)(\cos^2 x+\sin^2 x+2\sin x\cos x)\\&=\cos x+\sin x\\&(\cos x-\sin x)(\cos x+\sin x)^2=\cos x+\sin x\\&(\cos^2 x-\sin^2 x)(\cos x+\sin x)=\cos x+\sin x\\&(\cos x-\sin x)(\cos 2x-1)=0\end{aligned}$$
$\cos x+\sin x=0$ or $\cos 2x=1$

$\tan x=-1$ or $2x=0^\circ,\ 360^\circ$

$x=135^\circ,\ 315^\circ$ or $x=0^\circ,\ 180^\circ$

$\therefore\ x=\underline{\underline{0^\circ,\ 135^\circ,\ 180^\circ,\ 315^\circ}}$

21.
$$\begin{aligned}\sin^4 x+\cos^4 x&=\frac{1}{2}\sin 2x\\(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x&=\frac{1}{2}\sin 2x\\1-\frac{1}{2}\sin^2 2x&=\frac{1}{2}\sin 2x\\\sin^2 2x+\sin 2x-2&=0\\(\sin 2x-1)(\sin 2x+2)&=0\end{aligned}$$
$\sin 2x=1$ or $\sin 2x=-2$ (rejected)

$2x=90^\circ,\ 450^\circ$

$x=\underline{\underline{45^\circ,\ 225^\circ}}$

22.
$$\begin{aligned}\cos 4x+\sin 2x&=0\\(2\cos^2 2x-1)+\sin 2x&=0\\(1-2\sin^2 2x)+\sin 2x&=0\\2\sin^2 2x-\sin 2x-1&=0\\(2\sin 2x+1)(\sin 2x-1)&=0\end{aligned}$$
$\sin 2x=-\dfrac{1}{2}$ or $\sin 2x=1$

$2x=210^\circ,\ 330^\circ,\ 570^\circ,\ 690^\circ$ or $2x=90^\circ,\ 450^\circ$

$x=\underline{\underline{45^\circ,\ 105^\circ,\ 165^\circ,\ 225^\circ,\ 285^\circ,\ 345^\circ}}$

23.
$$\begin{aligned}\frac{1+\cos 2x}{2\cos x}&=\frac{\sin 2x}{1-\cos 2x}\\\frac{1+\cos 2x}{2\cos x}-\frac{\sin 2x}{1-\cos 2x}&=0\\\frac{2\cos^2 x}{2\cos x}-\frac{2\sin x\cos x}{2\sin^2 x}&=0\\\frac{\sin^2 x\cos^2 x-\sin x\cos^2 x}{\cos x\sin^2 x}&=0\\\frac{\sin x\cos^2 x(\sin x-1)}{\cos x\sin^2 x}&=0\end{aligned}$$
$\therefore\ \sin x=1$ for $\sin x\neq 0,\ \cos x\neq 0$

$x=\underline{\underline{90^\circ}}$

24. (a)
$$\begin{aligned}y&=\sec^2\theta\csc^2\theta-2\\&=\frac{1}{\sin^2\theta\cos^2\theta}-2\\&=\frac{4}{4\sin^2\theta\cos^2\theta}-2\\&=\frac{4}{\sin^2 2\theta}-2\end{aligned}$$

(b) y is minimum when $\sin^2 2\theta = 1$.

$\therefore$ The minimum value of y is

$$\frac{4}{1} - 2 = 4 - 2 = \underline{\underline{2}}$$

25. $\frac{2\tan x}{1-\tan^2 x} = \tan 2x = \tan 45° = 1$

$$\therefore \quad 2\tan x = 1 - \tan^2 x$$
$$\tan^2 x + 2\tan x - 1 = 0$$
$$\tan x = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1}$$
$$= \frac{-2 \pm \sqrt{8}}{2}$$
$$= -1 \pm \sqrt{2}$$
$$\tan 22.5° = \underline{\underline{\sqrt{2}-1}} \ (\because \text{positive})$$

26. (a) Let $y = \cos\theta\cos 2\theta\cos 4\theta$

$$\therefore \quad y\sin\theta = \sin\theta\cos\theta\cos 2\theta\cos 4\theta$$
$$= \frac{1}{2}\sin 2\theta\cos 2\theta\cos 4\theta$$
$$= (\frac{1}{2})^2 \sin 4\theta\cos 4\theta$$
$$= (\frac{1}{2})^3 \sin 8\theta$$
$$\therefore \quad y = \frac{1}{8}\sin 8\theta\csc\theta$$

(b) Let $\theta = 20°$,

$$\therefore \quad y = \frac{1}{8}\sin 160° \csc 20°$$
$$= \frac{1}{8}\cdot\frac{\sin(180° - 20°)}{\sin 20°}$$
$$= \frac{1}{8}\cdot\frac{\sin 20°}{\sin 20°}$$
$$= \underline{\underline{\frac{1}{8}}}$$

27. (a) $y = \sin^2 x + \sin x\cos x$

$$= \frac{1}{2}(1 - \cos 2x) + \frac{1}{2}\sin 2x$$
$$= \frac{1}{2}(1 + \sin 2x - \cos 2x)$$
$$= \frac{1}{2} + \frac{1}{2}\sin 2x - \frac{1}{2}\cos 2x$$
$$p + r\sin(2x - \alpha)$$
$$= p + r\sin 2x\cos\alpha - r\sin\alpha\cos 2x$$
$$p = \frac{1}{2}$$
$$\begin{cases} r\cos\alpha = \frac{1}{2} & \text{..................(1)} \\ -r\sin\alpha = -\frac{1}{2} & \text{..............(2)} \end{cases}$$

$$\frac{(2)}{(1)}, \quad \frac{\sin\alpha}{\cos\alpha} = 1$$
$$\tan\alpha = 1,$$
$$\alpha = \frac{\pi}{4} \ (\alpha \text{ is an acute angle})$$
$$(1)^2 + (2)^2,$$
$$r^2\cos^2\alpha + r^2\sin^2\alpha = \frac{1}{4} + \frac{1}{4}$$
$$r^2(\cos^2\alpha + \sin^2\alpha) = \frac{1}{2}$$
$$r = \pm\sqrt{\frac{1}{2}}$$
$$= \sqrt{\frac{1}{2}} \text{ or } -\sqrt{\frac{1}{2}} \text{ (rejected)}$$
$$\therefore \quad y = \frac{1}{2} + \frac{1}{\sqrt{2}}\sin(2x - \alpha)$$
$$= \underline{\underline{\frac{1}{2} + \frac{\sqrt{2}}{2}\sin(2x - \frac{\pi}{4})}}$$

(b) y is maximum when $\sin(2x - \alpha)$ is maximum.

y is minimum when $\sin(2x - \alpha)$ is minimum.

$-1 \le \sin(2x - \alpha) \le 1$

$\therefore$ The maximum value of y

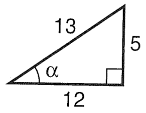
$$= \frac{1}{2} + \frac{\sqrt{2}}{2}(1) = \underline{\underline{\frac{1+\sqrt{2}}{2}}}$$

The minimum value of y

$$= \frac{1}{2} + \frac{\sqrt{2}}{2}(-1) = \underline{\underline{\frac{1-\sqrt{2}}{2}}}$$

Exercise 6C (p.156)

1. $12\cos\theta + 5\sin\theta$



$$= 13(\frac{12}{13}\cos\theta + \frac{5}{13}\sin\theta)$$
$$= 13(\cos\alpha\cos\theta + \sin\alpha\sin\theta)$$
$$= 13\cos(\theta - \alpha) \text{ where } \tan\alpha = \frac{5}{12}.$$

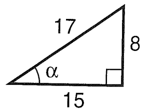
Since $-1 \le \cos(\theta - \alpha) \le 1$

$\therefore \quad -13 \le 13\cos(\theta - \alpha) \le 13$

$\therefore$ The maximum value is $\underline{\underline{13}}$.

The minimum value is $\underline{\underline{-13}}$.

2. $15\sin\theta - 8\cos\theta$



$$= 17(\frac{15}{17}\sin\theta - \frac{8}{17}\cos\theta)$$
$$= 17(\cos\alpha\sin\theta - \sin\alpha\cos\theta)$$
$$= 17\sin(\theta - \alpha) \text{ where } \tan\alpha = \frac{8}{15}.$$

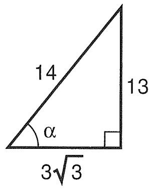
Since $-1 \le \sin(\theta - \alpha) \le 1$

$\therefore \quad -17 \le 17\sin(\theta - \alpha) \le 17$

$\therefore$ The maximum value is $\underline{\underline{17}}$.

The minimum value is $\underline{\underline{-17}}$.

3. $5\cos\theta + 3\cos(\theta + 60°)$



$$= 5\cos\theta + 3\cos\theta\cos 60° - 3\sin\theta\sin 60°$$
$$= 5\cos\theta + \frac{3}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$$
$$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$$
$$= \frac{14}{2}(\frac{13}{14}\cos\theta - \frac{3\sqrt{3}}{14}\sin\theta)$$
$$= \frac{14}{2}(\sin\alpha\cos\theta - \cos\alpha\sin\theta)$$
$$= 7\sin(\alpha - \theta) \text{ where } \tan\alpha = \frac{13}{3\sqrt{3}}.$$

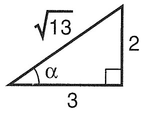
Since $-1 \le \sin(\alpha - \theta) \le 1$

$\therefore \quad -7 \le 7\sin(\alpha - \theta) \le 7$

$\therefore$ The maximum value is $\underline{\underline{7}}$.

The minimum value is $\underline{\underline{-7}}$.

4. $(2\cos\theta + 3\sin\theta)^2$



$$= [\sqrt{13}(\frac{2}{\sqrt{13}}\cos\theta + \frac{3}{\sqrt{13}}\sin\theta)]^2$$
$$= [\sqrt{13}(\sin\alpha\cos\theta + \cos\alpha\sin\theta)]^2$$
$$= [\sqrt{13}\sin(\alpha + \theta)]^2$$
$$= 13\sin^2(\alpha + \theta) \text{ where } \tan\alpha = \frac{2}{3}.$$

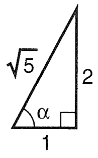
Since $0 \le \sin^2(\alpha + \theta) \le 1$

$\therefore \quad 0 \le 13\sin^2(\alpha + \theta) \le 13$

$\therefore$ The maximum value is $\underline{\underline{13}}$.

The minimum value is $\underline{\underline{0}}$.

5. $\sin\theta - 2\cos\theta$



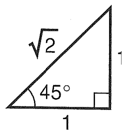
$$= \sqrt{5}(\frac{1}{\sqrt{5}}\sin\theta - \frac{2}{\sqrt{5}}\cos\theta)$$
$$= \sqrt{5}(\cos\alpha\sin\theta - \sin\alpha\cos\theta)$$
$$= \sqrt{5}\sin(\theta - \alpha) \text{ where } \tan\alpha = 2.$$
$$\therefore \quad \frac{1}{(\sin\theta - 2\cos\theta)^2} = \frac{1}{5\sin^2(\theta - \alpha)}$$

Since $\quad 0 \le \sin^2(\theta - \alpha) \le 1$

$$0 \le 5\sin^2(\theta - \alpha) \le 5$$
$$\frac{1}{5\sin^2(\theta - \alpha)} \ge \frac{1}{5}$$

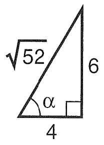
$\therefore$ The minimum value of $\frac{1}{(\sin\theta - 2\cos\theta)^2} = \underline{\underline{\frac{1}{5}}}$.

6.



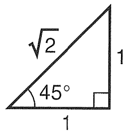
$$\cos x + \sin x = \sqrt{2}$$
$$\sqrt{2}(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x) = \sqrt{2}$$
$$\sqrt{2}(\cos 45°\cos x + \sin 45°\sin x) = \sqrt{2}$$
$$\cos(x - 45°) = 1$$
$$x - 45° = 0°$$
$$x = \underline{\underline{45°}}$$

7. $4\cos x + 6\sin x = 5$



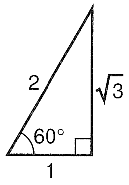
$$\sqrt{52}(\frac{4}{\sqrt{52}}\cos x + \frac{6}{\sqrt{52}}\sin x) = 5$$
$$\sqrt{52}(\cos\alpha\cos x + \sin\alpha\sin x) = 5$$
$$\cos(x - \alpha) = \frac{5}{\sqrt{52}} \text{ where } \tan\alpha = \frac{6}{4} = \frac{3}{2}.$$
$$x - 56.31°$$
$$= -46.102°, \ 46.102°$$
$$= \underline{\underline{10.21°, \ 102.4°}} \text{ (corr.to 4 sig. fig.)}$$

8. $2(\sin x + \cos x) = \sqrt{3} + 1$



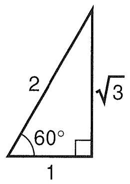
$$2\sqrt{2}(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x)$$
$$= \sqrt{3} + 1$$
$$2\sqrt{2}(\cos 45°\sin x + \sin 45°\cos x) = \sqrt{3} + 1$$
$$\sin(x + 45°) = \frac{\sqrt{3}+1}{2\sqrt{2}}$$
$$x + 45° = 75°, \ 105°$$
$$x = \underline{\underline{30°, \ 60°}}$$

9. $\sin x - \sqrt{3}\cos x = \sqrt{2}$



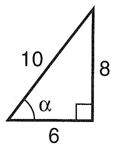
$$2(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x) = \sqrt{2}$$
$$2(\cos 60°\sin x - \sin 60°\cos x) = \sqrt{2}$$
$$2\sin(x - 60°) = \sqrt{2}$$
$$\sin(x - 60°) = \frac{\sqrt{2}}{2}$$
$$x - 60° = 45°, \ 135°$$
$$x = \underline{\underline{105°, \ 195°}}$$

10. $\cos x = \sqrt{3}(1 - \sin x)$



$$\cos x + \sqrt{3}\cos x = \sqrt{3}$$
$$2(\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x) = \sqrt{3}$$
$$2(\cos 60°\cos x + \sin 60°\sin x) = \sqrt{3}$$
$$\cos(x - 60°) = \frac{\sqrt{3}}{2}$$
$$x - 60° = -30°, \ 30°$$
$$x = \underline{\underline{30°, \ 90°}}$$

11. $\dfrac{2\sin x-1}{1+4\cos x}=\dfrac{2}{3}$



$6\sin x-3=2+8\cos x$

$6\sin x-8\cos x=5$

$10(\dfrac{6}{10}\sin x-\dfrac{8}{10}\cos x)=5$

$10(\cos\alpha\sin x-\sin\alpha\cos x)=5$

$\sin(x-\alpha)=\dfrac{1}{2}$ where $\tan\alpha=\dfrac{8}{6}=\dfrac{4}{3}$.

$x-53.10°=30°,\ 150°$

$x=\underline{\underline{83.13°,\ 203.1°}}$ (corr. to 4 sig. fig.)

12. (a) $r\cos(\theta-\alpha)=r(\cos\theta\cos\alpha+\sin\theta\sin\alpha)$

$=r\cos\theta\cos\alpha+r\sin\theta\sin\alpha$

As $5\cos\theta+12\sin\theta=r\cos(\theta-\alpha)$

$\therefore\ r\cos\alpha=5,\ r\sin\alpha=12$

$r=\sqrt{5^2+12^2}$

$=13$

$\tan\alpha=\dfrac{12}{5},\ \alpha=1.176$ (corr. to 4 sig. fig.)

$\therefore\ \underline{\underline{5\cos\theta+12\sin\theta=13\cos(\theta-1.176)}}$

(b) $f(\theta)$

$=(5\cos\theta+12\sin\theta)^2+10\cos\theta+24\sin\theta+1$

$=(5\cos\theta+12\sin\theta)^2+2(5\cos\theta+12\sin\theta)+1$

$=[(5\cos\theta+12\sin\theta)+1]^2$

$=[13\cos(\theta-1.176)+1]^2$

Since $[13\cos(\theta-1.176)+1]^2\geq 0$

$\therefore$ The minimum value of $f(\theta)=\underline{\underline{0}}$

$f(\theta)$ is maximum when $\cos(\theta-1.176)=1$

i.e. The maximum value of $f(\theta)$

$=[13(1)+1]^2=14^2=\underline{\underline{196}}$

Revision Exercise 6 (p.157)

1. $\tan(A+B)=\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=1$

$\therefore\ A+B=45°$

$\tan C=1,\ \therefore\ C=45°$

$A+B+C=90°$

$\cos(A+B+C)=\underline{\underline{0}}$

2. Since $\sin A=\dfrac{3}{5}$ and A is acute,

$\therefore\ \tan A=\dfrac{3}{4}$

$\tan B=\tan[(A+B)-A]$

$=\dfrac{\tan(A+B)-\tan A}{1+\tan(A+B)\tan A}$

$=\dfrac{\frac{24}{7}-\frac{3}{4}}{1+\frac{24}{7}\cdot\frac{3}{4}}$

$=\dfrac{96-21}{28+72}$

$=\dfrac{75}{100}$

$=\dfrac{3}{4}$

$=\tan A$

A and B are both acute and $\tan A=\tan B$,

$\therefore\ A=B$

3. $A+B+C=90°$

$A+B=90°-C$

$\tan(A+B)=\tan(90°-C)$

$\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\dfrac{1}{\tan C}$

$\tan C(\tan A+\tan B)=1-\tan A\tan B$

$\tan A\tan B+\tan B\tan C+\tan C\tan A=1$

4. No solution is provided for the H.K.C.E.E. question because of the copyright reasons.

5. $\sec^2 x=\dfrac{2-\cos x-\sin x}{1-\sin x}$ for $1-\sin x\neq 0$

$\dfrac{1}{\cos^2 x}=\dfrac{2-\cos x-\sin x}{1-\sin x}$

$\dfrac{1}{(1-\sin x)(1+\sin x)}=\dfrac{2-\cos x-\sin x}{1-\sin x}$

$(1+\sin x)(2-\cos x-\sin x)=1$

$\cos x-\sin x+\sin x\cos x+\sin^2 x=1$

$\cos x-\sin x+\sin x\cos x+(1-\cos^2 x)=1$

$\cos x(1-\cos x)-\sin x(1-\cos x)=0$

$(\cos x-\sin x)(1-\cos x)=0$

$\cos x-\sin x=0$ or $1-\cos x=0$

$\tan x=1$ or $\cos x=1$

$x=45°,\ 225°$ or $x=0°$

$\therefore\ x=\underline{\underline{0°,\ 45°,\ 225°}}$

6. Let $P(n)$ be the proposition

" $\cos 2\theta\cos 2^2\theta\cdots\cos 2^n\theta=\dfrac{\sin 2^{n+1}\theta}{2^n\sin 2\theta}$ ".

When $n=1$,

L.H.S. $=\cos 2\theta$

R.H.S. $=\dfrac{\sin 2^{1+1}\theta}{2\sin 2\theta}$

$=\dfrac{\sin 4\theta}{2\sin 2\theta}$

$=\dfrac{2\sin 2\theta\cos 2\theta}{2\sin 2\theta}$

$=\cos 2\theta$

$=$L.H.S.

$\therefore\ P(1)$ is true.

Assume $P(k)$ is true for any positive integer k.

i.e. $\cos 2\theta\cos 2^2\theta\cdots\cos 2^k\theta=\dfrac{\sin 2^{k+1}\theta}{2^k\sin 2\theta}$

Then $\cos 2\theta\cos 2^2\theta\cdots\cos 2^k\theta\cdot\cos 2^{k+1}\theta$

$=\dfrac{\sin 2^{k+1}\theta}{2^k\sin 2\theta}\cos 2^{k+1}\theta$

$=\dfrac{2\sin 2^{k+1}\theta\cos 2^{k+1}\theta}{2^{k+1}\sin 2\theta}$

$=\dfrac{\sin 2(2^{k+1}\theta)}{2^{k+1}\sin 2\theta}$

$=\dfrac{\sin 2^{k+2}\theta}{2^{k+1}\sin 2\theta}$

$=\dfrac{\sin 2^{(k+1)+1}\theta}{2^{k+1}\sin 2\theta}$

Thus assuming $P(k)$ is true for any positive integer k, $P(k+1)$ is also true. By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

7. $f(x)=\sin^6 x+\cos^6 x$

$=(\sin^2 x+\cos^2 x)$

$(\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x)$

$=\sin^4 x+2\sin^2 x\cos^2 x$

$+\cos^4 x-3\sin^2 x\cos^2 x$

$=(\cos^2 x+\sin^2 x)^2-3\sin^2 x\cos^2 x$

$=1-3\sin^2 x\cos^2 x$

$=1-\dfrac{3}{4}\sin^2 2x$

$=1-\dfrac{3}{8}(1-\cos 4x)$

$=\underline{\underline{\dfrac{5}{8}+\dfrac{3}{8}\cos 4x}}$

Since $-1\leq\cos 4x\leq 1$

$-\dfrac{3}{8}\leq\dfrac{3}{8}\cos 4x\leq\dfrac{3}{8}$

$-\dfrac{3}{8}+\dfrac{5}{8}\leq\dfrac{5}{8}+\dfrac{3}{8}\cos 4x\leq\dfrac{3}{8}+\dfrac{5}{8}$

$\dfrac{1}{4}\leq\dfrac{5}{8}+\dfrac{3}{8}\cos 4x\leq 1$

$\therefore$ The maximum value of $f(x)=\underline{\underline{1}}$.

The minimum value of $f(x)=\underline{\underline{\dfrac{1}{4}}}$.

8. (a) L.H.S. $=\cot\theta+\tan\theta$

$=\dfrac{1}{\tan\theta}+\tan\theta$

$=\dfrac{1+\tan^2\theta}{\tan\theta}$

$=\dfrac{\sec^2\theta}{\tan\theta}$

$=\dfrac{1}{\cos^2\theta}\cdot\dfrac{\cos\theta}{\sin\theta}$

$=\dfrac{1}{\sin\theta\cos\theta}$

$=\dfrac{2}{2\sin\theta\cos\theta}$

$=\dfrac{2}{\sin 2\theta}$

$=2\csc 2\theta$

$=$R.H.S.

(b) $(\cot\theta+\tan\theta)^2=4$

$(2\csc 2\theta)^2=4$ (by **(a)**)

$\csc^2 2\theta=1$

$(\csc^2 2\theta-1)=0$

$(\csc 2\theta+1)(\csc 2\theta-1)=0$

$\csc 2\theta=-1$ or $\csc 2\theta=1$

$\sin 2\theta=-1$ or $\sin 2\theta=1$

$2\theta=270°$ or $2\theta=90°$

$\theta=135°$ or $\theta=45°$

$\therefore\ \theta=\underline{\underline{45°,\ 135°}}$

9. (a) $\tan\dfrac{5\pi}{12}=\tan(\dfrac{\pi}{4}+\dfrac{\pi}{6})=\dfrac{1+\frac{1}{\sqrt{3}}}{1-1\cdot\frac{1}{\sqrt{3}}}=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

(b) $\sqrt{3}(\cos\theta+\sin\theta)+(\cos\theta-\sin\theta)$

$=R\sin(\theta+\phi)$

$(\sqrt{3}-1)\sin\theta+(\sqrt{3}+1)\cos\theta$

$=R(\sin\theta\cos\phi+\cos\theta\sin\phi)$

$=(R\cos\phi)\sin\theta+(R\sin\phi)\cos\theta$

$\therefore\ \begin{cases}R\cos\phi=\sqrt{3}-1\\ R\sin\phi=\sqrt{3}+1\end{cases}$

$R^2=(\sqrt{3}-1)^2+(\sqrt{3}+1)^2=8$

$\therefore\ R=\underline{\underline{2\sqrt{2}}}$

$\tan\phi=\dfrac{\sqrt{3}+1}{\sqrt{3}-1},\ \therefore\ \phi=\underline{\underline{\dfrac{5\pi}{12}}}$ (by **(a)**)

10. (a) $\cos x-\sqrt{3}\sin x\equiv r\cos(x+\alpha)$

$\cos x-\sqrt{3}\sin x\equiv r(\cos x\cos\alpha-\sin x\sin\alpha)$

$\cos x-\sqrt{3}\sin x\equiv r\cos x\cos\alpha-r\sin x\sin\alpha$

$r\cos\alpha=1,\ r\sin\alpha=\sqrt{3}$

$r^2=1^2+(\sqrt{3})^2=\underline{\underline{4}}$

$\tan\alpha=\underline{\underline{\sqrt{3}}}$

(b) $r=2$ or -2 (rejected)

$\tan\alpha=\sqrt{3},\ \alpha=60°$

$\cos x-\sqrt{3}\sin x=1$

$2\cos(x+60°)=1$

$\cos(x+60°)=\frac{1}{2}$

$x+60°=60°,\ -60°$

$x=\underline{\underline{0°,\ -120°}}$

(c) $\frac{1}{(\cos x-\sqrt{3}\sin x)^2}=\frac{1}{[2\cos(x+60°)]^2}$

$=\frac{1}{4\cos^2(x+60°)}$

$\frac{1}{(\cos x-\sqrt{3}\sin x)^2}$ is minimum

when $\cos^2(x+60°)=1$.

$\therefore$ The least value of $\frac{1}{(\cos x-\sqrt{3}\sin x)^2}$ is $\underline{\underline{\frac{1}{4}}}$.

11. (a) $6\cos x+8\sin x=r\cos(x-\alpha)$

$=r\cos x\cos\alpha+r\sin x\sin\alpha$

$\therefore\ \begin{cases}r\cos\alpha=6\\ r\sin\alpha=8\end{cases}$

$r^2=6^2+8^2=100$

$r=\underline{\underline{10}}$ or $r=-10$ (rejected)

$\tan\alpha=\frac{8}{6}=\frac{4}{3}$

$\alpha=\underline{\underline{53.13°}}$ (corr. to 2 d.p.)

(b) By **(a)**,

$6\cos x+8\sin x+13=10\cos(x-53.13°)+13$

$-1\le\cos(x-53.13°)\le1$

$-10\le10\cos(x-53.13°)\le10$

$-10+13\le10\cos(x-53.13°)+13\le10+13$

$3\le6\cos x+8\sin x+13\le23$

$\frac{1}{23}\le\frac{1}{6\cos x+8\sin x+13}\le\frac{1}{3}$

$\therefore\ \underline{\underline{\frac{1}{23}\le y\le\frac{1}{3}}}$

12. (a) $A+B+C=180°$

$A+B=180°-C$

$\tan(A+B)=\tan(180°-C)$

$\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$

$\tan A+\tan B=-\tan C+\tan A\tan B\tan C$

$\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(b) Let $\tan A=k,\ \tan B=2k,\ \tan C=-6k$,

then $k+2k-6k=(k)(2k)(-6k)$ (by **(a)**)

$-3k=-12k^3$

$12k^3-3k=0$

$3k(4k^2-1)=0$

$4k^2-1=0$

$k^2=\frac{1}{4}$

$k=\pm\frac{1}{2}$

When $k=\frac{1}{2}$, $\tan A=\frac{1}{2}$, $\tan B=1$, $\tan C=-3$,

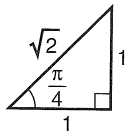
$A=\underline{\underline{26.6°}},\ B=\underline{\underline{45°}},\ C=\underline{\underline{108.4°}}$

When $k=-\frac{1}{2}$, A, B are both obtuse, which is impossible for any triangle.

13. (a) $\sin x+\cos x$

$=\sqrt{2}(\frac{1}{2}\cos x+\frac{1}{\sqrt{2}}\sin x)$

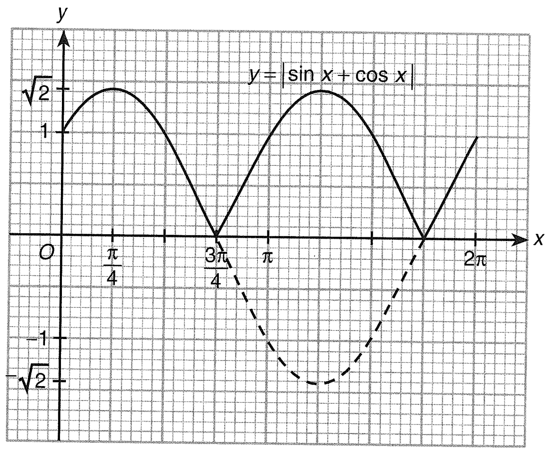
$=\underline{\underline{\sqrt{2}\cos(x-\frac{\pi}{4})}}$



(b) Let $y=f(x)$

$=|\sin x+\cos x|$

$=\sqrt{2}\left|\cos(x-\frac{\pi}{4})\right|$



(c) $|\sin x+\cos x|=1$

$\left|\cos(x-\frac{\pi}{4})\right|=\frac{1}{\sqrt{2}}$

$\cos(x-\frac{\pi}{4})=\frac{1}{\sqrt{2}}$ or $\cos(x-\frac{\pi}{4})=-\frac{1}{\sqrt{2}}$

$x-\frac{\pi}{4}=\frac{\pi}{4},\ -\frac{\pi}{4}$ or $x-\frac{\pi}{4}=\frac{3\pi}{4}$

$\therefore\ \underline{\underline{x=0,\ \frac{\pi}{2},\ \pi}}$

14. $4(\cos\theta-\sin\theta)+k\sin\theta=5$

$4\cos\theta+(k-4)\sin\theta=5$

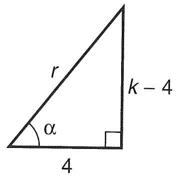
$r(\frac{4}{r}\cos\theta+\frac{k-4}{r}\sin\theta)=5$

$r(\cos\alpha\cos\theta+\sin\alpha\sin\theta)=5$

$r\cos(\theta-\alpha)=5$

$\cos(\theta-\alpha)=\frac{5}{r}$

$\cos^2(\theta-\alpha)=(\frac{5}{r})^2$



It has no solution, the condition is

$(\frac{5}{r})^2>1$

$5^2>r^2$

$(k-4)^2+4^2<5^2$

$16+k^2-8k+16<25$

$k^2-8k+7<0$

$(k-1)(k-7)<0$

$\underline{\underline{1<k<7}}$

15. $\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

$1=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

$1-\tan\theta\tan\phi=\tan\theta+\tan\phi$

$\tan\phi+\tan\theta\tan\phi=1-\tan\theta$

$\tan\phi=\underline{\underline{\frac{1-\tan\theta}{1+\tan\theta}}}$

$(1+\tan\theta)(1+\tan\theta)$

$=1+\tan\theta+\tan\phi+\tan\theta\tan\phi$

$=1+\tan\theta+(1-\tan\theta)$

$=2$

Let $\theta=\phi=\frac{\pi}{8}$,

$\therefore\ (1+\tan\frac{\pi}{8})^2=2$

$1+\tan\frac{\pi}{8}=\sqrt{2}$ (since $1+\tan\frac{\pi}{8}>0$)

$\tan\frac{\pi}{8}=\sqrt{2}-1$

16. (a) $A+B+C=\pi$

$\therefore\ A=\pi-(B+C)$

$\sin A=\sin[\pi-(B+C)]=\sin(B+C)$

(b) (i) $\tan B=\frac{\cos(B-C)}{\sin A-\sin(B-C)}$

$=\frac{\cos(B-C)}{\sin(B+C)-\sin(B-C)}$

$=\cos B\cos C+\sin B\sin C$

$\div[(\sin B\cos C+\sin C\cos B)$

$-(\sin B\cos C-\sin C\cos B)]$

$\therefore\ \frac{\sin B}{\cos B}=\frac{\cos B\cos C+\sin B\sin C}{2\cos B\sin C}$

$\cos^2B\cos C-\sin B\cos B\sin C=0$

$\cos B(\cos B\cos C-\sin B\sin C)=0$

$\cos B\cos(B+C)=0$

(ii) As $\cos B\cos(B+C)=0$

$\therefore\ \cos B=0$ or $\cos(B+C)=0$

As $\cos B\neq0$ (otherwise $\tan B$ undefined), $\cos(B+C)=0$.

Also, $0<B+C<\pi$, $\therefore\ B+C=\frac{\pi}{2}$

$\therefore\ A=\pi-(B+C)=\frac{\pi}{2}$

17. (a) $\cos(\theta+\alpha)=p$

$\cos\theta\cos\alpha-\sin\theta\sin\alpha=p$ ....(1

$\sin(\theta+\beta)=q$

$\sin\theta\cos\beta+\sin\beta\cos\theta=q$ ....(2)

(1) × $\cos\beta$, $\cos\theta\cos\alpha\cos\beta-\sin\theta\sin\alpha\cos$

$=p\cos\beta$ ....(3)

(2) × $\sin\alpha$, $\sin\alpha\sin\theta\cos\beta+\sin\alpha\sin\beta\cos$

$=q\sin\alpha$ ....(4)

(3) + (4), $\cos\theta\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\theta$

$=p\cos\beta+q\sin\alpha$

$\cos\theta(\cos\alpha\cos\beta+\sin\alpha\sin\beta)$

$=p\cos\beta+q\sin\alpha$

$\cos\theta=\underline{\underline{\frac{p\cos\beta+q\sin\alpha}{\cos(\alpha-\beta)}}}$

(1) × $\sin\beta$, $\sin\beta\cos\theta\cos\alpha-\sin\beta\sin\theta\sin\alpha$

$=p\sin\beta$ ....(5)

(2) × $\cos\alpha$, $\cos\alpha\sin\theta\cos\beta+\cos\alpha\sin\beta\cos$

$=q\cos\alpha$ ....(6)

(6) – (5), $\cos\alpha\sin\theta\cos\beta+\sin\beta\sin\theta\sin\alpha$

$=q\cos\alpha-p\sin\beta$

$\sin\theta(\cos\alpha\cos\beta+\sin\alpha\sin\beta)$

$=q\cos\alpha-p\sin\beta$

$\sin\theta=\underline{\underline{\frac{q\cos\alpha-p\sin\beta}{\cos(\alpha-\beta)}}}$

(b) $\sin^2\theta+\cos^2\theta=1$

$(q\cos\alpha-p\sin\beta)^2+(p\cos\beta+q\sin\alpha)^2$

$=\cos^2(\alpha-\beta)$

$q^2\cos^2\alpha-2pq\sin\beta\cos\alpha+p^2\sin^2\beta$

$+p^2\cos^2\beta+2pq\sin\alpha\cos\beta+q^2\sin^2\alpha$

$=\cos^2(\alpha-\beta)$

$\therefore\quad p^2+q^2+2pq(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$

$=\cos^2(\alpha-\beta)$

$p^2+q^2+2pq\sin(\alpha-\beta)=\cos^2(\alpha-\beta)$

18. (a) $(\sin x+\cos x+1)(\sin x+\cos x-1)$

$=\sin^2x+2\sin x\cos x+\cos^2x-1$

$=2\sin x\cos x$

(b) (i) $f(x)$

$=\sec x+\csc x+\sec x\csc x$

$=\dfrac{1}{\cos x}+\dfrac{1}{\sin x}+\dfrac{1}{\cos x\sin x}$

$=\dfrac{\sin x+\cos x+1}{\cos x\sin x}$

$=[(\sin x+\cos x+1)\cdot(\sin x+\cos x-1)]$

$\div[\cos x\sin x(\sin x+\cos x-1)]$

$=\dfrac{2\sin x\cos x}{\cos x\sin x(\sin x+\cos x-1)}$

$=\dfrac{2}{\sin x+\cos x-1}$

(ii) $f(x)=\dfrac{2}{\sin x+\cos x-1}$

$=\dfrac{2}{\sqrt{2}\sin(x+\frac{\pi}{4})-1}$

When $0<x<\dfrac{\pi}{2}$, $1\le\sqrt{2}\sin(x+\dfrac{\pi}{4})\le\sqrt{2}$

$\therefore$ $f(x)$ is minimum when $x=\dfrac{\pi}{4}$.

The minimum value of $f(x)=\underline{\underline{\dfrac{2}{\sqrt{2}-1}}}$

19. No solution is provided for the H.K.C.E.E. question because of the copyright reasons.

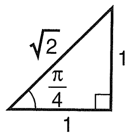
20. (a) $f(\theta)$

$=\sin\theta+\cos\theta$

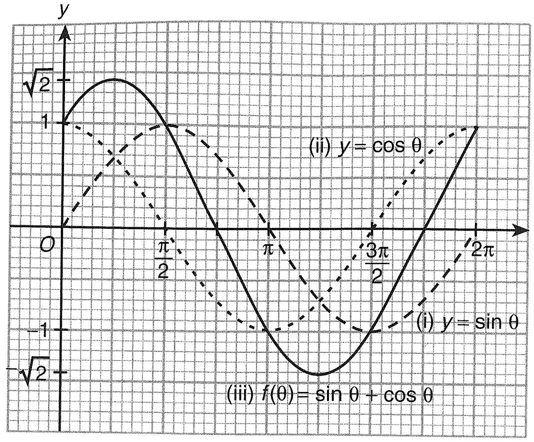
$=\sqrt{2}(\dfrac{1}{\sqrt{2}}\sin\theta+\dfrac{1}{\sqrt{2}}\cos\theta)$

$=\sqrt{2}(\sin\dfrac{\pi}{4}\sin\theta+\cos\dfrac{\pi}{4}\cos\theta)$

$=\underline{\underline{\sqrt{2}\cos(\theta-\dfrac{\pi}{4})}}$



(b)



(c) (i) $y=4\sin\theta+4\cos\theta-3$

$=4(\sin\theta+\cos\theta)-3$

$=4\sqrt{2}\cos(\theta-\dfrac{\pi}{4})-3$

$\because\quad\cos(\theta-\dfrac{\pi}{4})\le1$

$4\sqrt{2}\cos(\theta-\dfrac{\pi}{4})\le4\sqrt{2}$

$4\sqrt{2}\cos(\theta-\dfrac{\pi}{4})-3\le4\sqrt{2}-3$

$\therefore$ The maximum value of y is $\underline{\underline{4\sqrt{2}-3}}$.

(ii) When y attains its maximum,

$\cos(\theta-\dfrac{\pi}{4})=1$

$\theta-\dfrac{\pi}{4}=0$

$\theta=\underline{\underline{\dfrac{\pi}{4}}}$

Enrichment 6 (p.160)

1. (a) $\cos(A+B+C)$

$=\cos[A+(B+C)]$

$=\cos A\cos(B+C)-\sin A\sin(B+C)$

$=\cos A(\cos B\cos C-\sin B\sin C)$

$-\sin A(\sin B\cos C+\sin C\cos B)$

$=\cos A\cos B\cos C-\sin A\sin B\cos C$

$-\sin A\cos B\sin C-\cos A\sin B\sin C$

(b) If $A+B+C=90°$,

$\cos90°=\cos A\cos B\cos C-\sin A\sin B\cos C$

$-\sin A\cos B\sin C-\cos A\sin B\sin C$

$0=\cos A\cos B\cos C-\sin A\sin B\cos C$

$-\sin A\cos B\sin C-\cos A\sin B\sin C$

$\cos A\cos B\cos C$

$=\sin A\sin B\cos C$

$+\sin A\cos B\sin C+\cos A\sin B\sin C$

2. (a) $\tan\theta+\cot\theta=\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}$

$=\dfrac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$

$=\dfrac{2}{2\sin\theta\cos\theta}$

$=\dfrac{2}{\sin2\theta}$

(b) Let x_1 and x_2 be the roots of the equation.

$\therefore\quad x_1+x_2=\tan\theta+\cot\theta$

$x_1x_2=1$

As $2-\sqrt{3}$ is a root, let $x_1=2-\sqrt{3}$.

$\therefore\quad x_2=\dfrac{1}{2-\sqrt{3}}=\dfrac{2+\sqrt{3}}{4-3}=2+\sqrt{3}$

$\therefore\quad x_1+x_2=4=\tan\theta+\cot\theta$

$=\dfrac{2}{\sin2\theta}$

$\therefore\quad\sin2\theta=\underline{\underline{\dfrac{1}{2}}}$

$\cos4\theta=1-2\sin^22\theta=1-\dfrac{2}{4}=\underline{\underline{\dfrac{1}{2}}}$

3. (a) $AB=BE$ (given)

$\angle BAE=\angle BEA$ (base ∠s, isos Δ)

$=\dfrac{1}{2}(180°-36°)$

$=72°$

$AC=AE$ (given)

$\angle ACE=\angle BEA$ (base ∠s, isos Δ)

$=72°$

$\angle ABC+\angle BAC=\angle ACE$ (ext. ∠ of Δ)

$\angle BAC=72°-36°$

$=\underline{\underline{36°}}$

(b) $BD=BC+CD=BC+AC\cos72°$

$=AC+AC\cos72°$

$=AC(1+\cos72°)$

$=AE[1+\cos(90°-72°)]$

$=y(1+\sin18°)$

(c) $BC=AC=AE=y$

$CE=2DE=2y\cos72°=2y\sin18°$

$AB=BE=BC+CE$

$=y+2y\sin18°$

$=y(1+2\sin18°)$

(d) In ΔABD, $\cos36°=\dfrac{BD}{AB}$

$=\dfrac{y(1+\sin18°)}{y(1+2\sin18°)}$

$=\dfrac{1+\sin18°}{1+2\sin18°}$

(e) From **(d)**,

$\cos36°=\dfrac{1+\sin18°}{1+2\sin18°}$

$1-2\sin^218°=\dfrac{1+\sin18°}{1+2\sin18°}$

Let $s=\sin18°$,

$(1-2s^2)(1+2s)=1+s$

$1+2s-2s^2-4s^3=1+s$

$s-2s^2-4s^3=0$

$4s^2+2s-1=0\quad(\because\ s\ne0)$

$\therefore\quad4\sin^218°+2\sin18°-1=0$

(f) From **(e)**,

$4s^2+2s-1=0$

$s=\dfrac{-2\pm\sqrt{(2)^2-4\cdot4(-1)}}{2(4)}$

$=\dfrac{-1\pm\sqrt{5}}{4}$

$\sin18°=\underline{\underline{\dfrac{\sqrt{5}-1}{4}}}$ or $\dfrac{-\sqrt{5}-1}{4}$ (rejected)

4. (a) (i) $OM=OP\cos\theta=\underline{10\cos\theta}$

$ON=OP\cos(\dfrac{\pi}{3}-\theta)=\underline{\underline{10\cos(\dfrac{\pi}{3}-\theta)}}$

(ii) $OM+ON$

$=10\cos\theta+10\cos(\dfrac{\pi}{3}-\theta)$

$=10[\cos\theta+\cos(\dfrac{\pi}{3}-\theta)]$

$=10(\cos\theta+\cos\dfrac{\pi}{3}\cos\theta+\sin\dfrac{\pi}{3}\sin\theta)$

$=10(\cos\theta+\dfrac{1}{2}\cos\theta+\dfrac{\sqrt{3}}{2}\sin\theta)$

$=10(\dfrac{3}{2}\cos\theta+\dfrac{\sqrt{3}}{2}\sin\theta)$

Let $r\sin\alpha=\dfrac{3}{2}$, $r\cos\alpha=\dfrac{\sqrt{3}}{2}$

$r^2=(\dfrac{3}{2})^2+(\dfrac{\sqrt{3}}{2})^2$

$=\dfrac{9}{4}+\dfrac{3}{4}$

$=\dfrac{12}{4}$

$=3$

$r=\sqrt{3}$

$r\sin\alpha=\sqrt{3}\sin\alpha=\dfrac{3}{2}$, $\sin\alpha=\dfrac{\sqrt{3}}{2}$

$r\cos\alpha = \sqrt{3}\cos\alpha = \frac{\sqrt{3}}{2}, \ \cos\alpha = \frac{1}{2}$

$\therefore \quad \alpha = \frac{\pi}{3}$

$\therefore \quad OM + ON$

$= 10\sqrt{3}(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$

$= 10\sqrt{3}\sin(\alpha + \theta)$

$= 10\sqrt{3}\sin(\frac{\pi}{3} + \theta)$

As $0 \le \theta \le \frac{\pi}{3}, \ \frac{\sqrt{3}}{2} \le \sin(\frac{\pi}{3} + \theta) \le 1$

$\therefore$ $OM + ON$ is maximum when

$\sin(\frac{\pi}{3} + \theta) = 1.$

i.e. $\theta = \frac{\pi}{6}$

Maximum of $(OM + ON) = \underline{\underline{10\sqrt{3}}}$

(b) (i) $PN = 10\sin(\frac{\pi}{3} - \theta)$

$\underline{\underline{\hphantom{PN = 10\sin(\frac{\pi}{3} - \theta)}}}$

$PM = \underline{\underline{10\sin\theta}}$

(ii) $PN + PM$

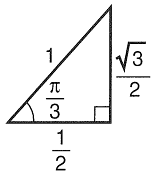
$= 10\sin(\frac{\pi}{3} - \theta) + 10\sin\theta$

$= 10(\sin\frac{\pi}{3}\cos\theta - \sin\theta\cos\frac{\pi}{3}) + 10\sin\theta$

$= 10(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta) + 10\sin\theta$

$= 10(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta)$

$= 10\sin(\frac{\pi}{3} + \theta)$



As $0 \le \theta \le \frac{\pi}{3}$,

$\therefore$ $PN + PM$ is maximum when

$\sin(\frac{\pi}{3} + \theta) = 1.$

i.e. $\theta = \frac{\pi}{6}$

Maximum of $(PN + PM) = \underline{\underline{10}}$

Classwork 1 (p.145)

1. (a) $\cos 15^\circ = \cos(45^\circ - 30^\circ)$

$= \cos 45^\circ\cos 30^\circ + \sin 45^\circ\sin 30^\circ$

$= (\frac{\sqrt{2}}{2})(\frac{\sqrt{3}}{2}) + (\frac{\sqrt{2}}{2})(\frac{1}{2})$

$= \underline{\underline{\frac{\sqrt{6} + \sqrt{2}}{4}}}$

(b) $\tan 75^\circ = \tan(45^\circ + 30^\circ)$

$= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ\tan 30^\circ}$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - (1)(\frac{1}{\sqrt{3}})}$

$= \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$

$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$

$= \frac{(\sqrt{3} + 1)^2}{2}$

$= \underline{\underline{2 + \sqrt{3}}}$

(c) $\cos 105^\circ = \cos(60^\circ + 45^\circ)$

$= \cos 60^\circ\cos 45^\circ - \sin 60^\circ\sin 45^\circ$

$= (\frac{1}{2})(\frac{\sqrt{2}}{2}) - (\frac{\sqrt{3}}{2})(\frac{\sqrt{2}}{2})$

$= \underline{\underline{\frac{\sqrt{2} - \sqrt{6}}{4}}}$

2. (a) $\sin 2A\cos A - \cos 2A\sin A$

$= \sin(2A - A)$

$= \underline{\underline{\sin A}}$

(b) $\sin A\sin B - \cos A\cos B$

$= -(\cos A\cos B - \sin A\sin B)$

$= \underline{\underline{-\cos(A + B)}}$

(c) $\sin(\theta - \phi)\cos\phi + \cos(\theta - \phi)\sin\phi$

$= \sin[(\theta - \phi) + \phi]$

$= \underline{\underline{\sin\theta}}$

(d) $\frac{\cos^2 A - \sin^2 A}{\cos 3A\cos A + \sin 3A\sin A}$

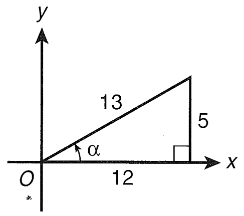
$= \frac{\cos^2 A - \sin^2 A}{\cos(3A - A)}$

$= \frac{\cos^2 A - \sin^2 A}{\cos 2A}$

$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A - \sin^2 A}$

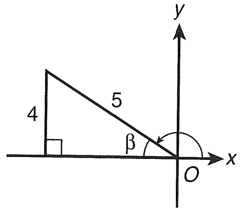
$= \underline{\underline{1}}$

3. $\sin\alpha = \frac{5}{13}, \ \cos\alpha = \frac{12}{13}, \ \tan\alpha = \frac{5}{12} \ (0 < \alpha < \frac{\pi}{2})$



$\sin\beta = \frac{4}{5}, \ \cos\beta = -\frac{3}{5},$

$\tan\beta = -\frac{4}{3} \ (\frac{\pi}{2} < \beta < \pi)$



(a) $\cos\alpha = \underline{\underline{\frac{12}{13}}}, \ \sin\beta = \underline{\underline{\frac{4}{5}}}$

(b) $\cos(\alpha - \beta)$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= (\frac{12}{13})(-\frac{3}{5}) + (\frac{5}{13})(\frac{4}{5})$

$= \underline{\underline{-\frac{16}{65}}}$

(c) $\tan(\alpha - \beta)$

$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$= \frac{\frac{5}{12} - (-\frac{4}{3})}{1 + (\frac{5}{12})(-\frac{4}{3})}$

$= \underline{\underline{\frac{63}{16}}}$

Classwork 2 (p.147)

1. $\sin(30^\circ + x) + \cos(60^\circ + x) - \cos x$

$= \sin 30^\circ\cos x + \sin x\cos 30^\circ$

$+ \cos 60^\circ\cos x - \sin 60^\circ\sin x - \cos x$

$= \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$

$- \frac{\sqrt{3}}{2}\sin x - \cos x$

$= 0$

2. $\cos^2(A - B) - \cos^2(A + B)$

$= [\cos(A - B) - \cos(A + B)]$

$[\cos(A - B) + \cos(A + B)]$

$= [(\cos A\cos B + \sin A\sin B)$

$- (\cos A\cos B - \sin A\sin B)]$

$[(\cos A\cos B + \sin A\sin B)$

$+ (\cos A\cos B - \sin A\sin B)]$

$= (2\sin A\sin B)(2\cos A\cos B)$

$= 4\sin A\sin B\cos A\cos B$

$= (2\sin A\cos A)(2\sin B\cos B)$

$= \sin 2A\sin 2B$

3. $\sin 2\theta\cos\theta + \cos 2\theta\sin\theta = \frac{1}{2}$

$\sin(2\theta + \theta) = \frac{1}{2}$

$\sin 3\theta = \frac{1}{2}$

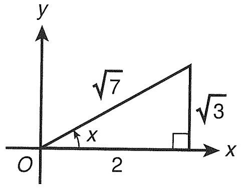
$3\theta = 30^\circ, \ 150^\circ, \ 390^\circ, \ 51$

$\theta = \underline{\underline{10^\circ, \ 50^\circ, \ 130^\circ, \ 170^\circ}}$

Classwork 3 (p.152)

1. $\tan x = \frac{\sqrt{3}}{2}, \ \cos x = \frac{2}{\sqrt{7}} = \frac{2\sqrt{7}}{7},$

$\sin x = \sqrt{\frac{3}{7}} = \frac{\sqrt{21}}{7}$



(a) $\cos x = \underline{\underline{\frac{2\sqrt{7}}{7}}}$

(b) $\cos 2x = \cos^2 x - \sin^2 x$

$= (\frac{2\sqrt{7}}{7})^2 - (\frac{\sqrt{21}}{7})^2$

$= \frac{7}{49}$

$= \underline{\underline{\frac{1}{7}}}$

(c) $\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2(\frac{\sqrt{3}}{2})}{1 - (\frac{\sqrt{3}}{2})^2} = \underline{\underline{4\sqrt{3}}}$

(d) $\tan 3x = \tan(2x + x)$

$= \frac{\tan 2x + \tan x}{1 - \tan 2x\tan x}$

$= \frac{4\sqrt{3} + (\frac{\sqrt{3}}{2})}{1 - (4\sqrt{3})(\frac{\sqrt{3}}{2})}$

$= \underline{\underline{-\frac{9\sqrt{3}}{10}}}$

2. (a) $\frac{\sin 2A}{1 - \cos 2A} = \frac{2\sin A\cos A}{1 - (1 - 2\sin^2 A)}$

$= \frac{2\sin A\cos A}{2\sin^2 A}$

$= \frac{\cos A}{\sin A}$

$= \cot A$

(b) $\tan 2A - 2\tan A$

$$= \frac{2\tan A}{1-\tan^2 A} - 2\tan A$$

$$= \frac{2\tan A - 2\tan A(1-\tan^2 A)}{1-\tan^2 A}$$

$$= \frac{2\tan A\tan^2 A}{1-\tan^2 A}$$

$$= \tan^2 A \cdot \frac{2\tan A}{1-\tan^2 A}$$

$$= \tan^2 A \tan 2A$$

3. $\sin x\cos x = \frac{\sqrt{3}}{4} \qquad 0° \le x < 360°$

$$\frac{1}{2}\sin 2x = \frac{\sqrt{3}}{4}$$

$$\sin 2x = \frac{\sqrt{3}}{2}$$

$$2x = 60°,\ 120°,\ 420°,\ 480°$$

$$x = \underline{\underline{30°,\ 60°,\ 210°,\ 240°}}$$

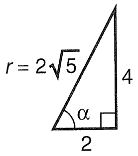
Classwork 4 (p.155)

1. $r\sin(\theta+\alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

$$= r\sin\theta\cos\alpha + r\cos\theta\sin\alpha$$

$$= 2\sin\theta + 4\cos\theta$$

Let $r\cos\alpha = 2,\ r\sin\alpha = 4$.



$\therefore\quad r = \sqrt{2^2+4^2}$

$$= 2\sqrt{5}$$

$$\tan\alpha = \frac{4}{2} = 2$$

$\alpha = 63.43°$ (corr. to 2 d.p.)

$\therefore\quad 2\sin\theta + 4\cos\theta = \underline{\underline{2\sqrt{5}\sin(\theta+63.43°)}}$

2. Let $r > 0$ and $0 \le \alpha < 90°$.

$r\cos(\theta-\alpha) = r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$$= r\cos\theta\cos\alpha + r\sin\theta\sin\alpha$$

As $5\sin\theta + 12\cos\theta = r\cos(\theta-\alpha)$

$\therefore\quad r\sin\alpha = 5,\ r\cos\alpha = 12$

$$r = \sqrt{5^2+12^2}$$

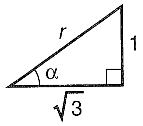
$$= 13$$

$$\tan\alpha = \frac{5}{12}$$

$\alpha = 22.62°$ (corr. to 2 d.p.)

$\therefore\quad 5\sin\theta + 12\cos\theta = \underline{\underline{13\cos(\theta-22.62°)}}$

3. Consider a right-angled triangle



$$r^2 = 1+3$$

$\therefore\quad r = 2$

and $\alpha = \tan^{-1}(\frac{1}{\sqrt{3}})$

$$= 30°$$

$\therefore\quad 2\sin 30° = 1$

$$2\cos 30° = \sqrt{3}$$

$\therefore$ The equation becomes

$$2\cos 30°\cos\theta - 2\sin 30°\sin\theta = 1$$

$$2\cos(30°+\theta) = 1$$

$$\cos(30°+\theta) = \frac{1}{2}$$

$$30°+\theta = 60°,\ 300°$$

$$\theta = \underline{\underline{30°,\ 270°}}$$

4. (a) $r > 0,\ 0 \le \alpha < \frac{\pi}{2}$

$r\sin(\theta-\alpha) = r(\sin\theta\cos\alpha - \sin\alpha\cos\theta)$

$$= r\sin\theta\cos\alpha - r\sin\alpha\cos\theta$$

As $\sqrt{3}\sin\theta - \cos\theta = r\sin(\theta-\alpha)$

$\therefore\quad r\cos\alpha = \sqrt{3},\ r\sin\alpha = 1$

$$r = \sqrt{3+1} = 2$$

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$\therefore\quad \sqrt{3}\sin\theta - \cos\theta = \underline{\underline{2\sin(\theta-\frac{\pi}{6})}}$

(b) Let $y = \sqrt{3}\sin\theta - \cos\theta = 2\sin(\theta-\frac{\pi}{6})$

Since $-1 \le \sin(\theta-\frac{\pi}{6}) \le 1$

$\therefore\quad -2 \le 2\sin(\theta-\frac{\pi}{6}) \le 2$

$\therefore$ The maximum value of y is $\underline{\underline{2}}$.

The minimum value of y is $\underline{\underline{-2}}$.

(c) $\sqrt{3}\sin\theta - \cos\theta = 1$

$$2\sin(\theta-\frac{\pi}{6}) = 1$$

$$\sin(\theta-\frac{\pi}{6}) = \frac{1}{2}$$

$$\theta - \frac{\pi}{6} = \frac{\pi}{6},\ \frac{5\pi}{6}$$

$$\theta = \underline{\underline{\frac{\pi}{3},\ \pi}}$$

CHAP

Exercis

1. 2sin

2. 2co

3. 4co

4. sin

5. cos

6. 4cos

7. cos60

8. sin5

9. cos2

10. sin(3

11. sin(

12. sin