7 Simple Harmonic Motion

Self Evaluation Exercise 7.3 (p.260)

1. D

For SHM,

$$a = -\omega^2 x$$

Since ω^2 is a positive constant, in the graph a against x, it is a straight line with a negative slope.

2. C

For SHM, "n complete oscillations in one second" means that the frequency of the motion is n Hz.

Angular frequency = $2\pi \times$ frequency = $2\pi n$ rad s^{-1}

3. C

$$\frac{d^2x}{dt^2} + Ax = 0$$

$$\frac{d^2x}{dt^2} = -Ax$$

$\because$ $\frac{d^2x}{dt^2}$ is the acceleration, the equation can be written as $a = -Ax$

As $a = -\omega^2 x$

$\therefore$ $\omega^2 = A$

$$\left(\frac{2\pi}{T}\right)^2 = A$$

$$T = \frac{2\pi}{\sqrt{A}}$$

4. D

For SHM,

$$A = -\omega^2 x$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$\therefore$ The solution of x can be represented as $x = A\sin(\omega t + B)$ where A is the amplitude and B is the initial angular displacement.

Then $v = \frac{dx}{dt} = \omega A\cos(\omega t + B)$

Since $-1 \le \cos(\omega t + B) \le 1$,

Maximum speed = ωA

$$= \left(\frac{2\pi}{0.1}\right)(2.0 \times 10^{-3})$$

$$= 1.3 \times 10^{-1}\text{ m s}^{-1}$$

5. C

$$x = A\sin(\omega t)$$

$$= A\sin\frac{2\pi t}{T}$$

For $t = \frac{T}{8}$,

$$x = A\sin\frac{\pi}{4}$$

$$= \frac{A}{\sqrt{2}}$$

6. C

The acceleration a is the rate of change of velocity. From the a-x graph, it can be found that when x is negative, a is positive but decreasing. a becomes 0 when $x = 0$ and keeps on decreasing when x is positive. Therefore, the velocity v increases and remains unchanged at $x = 0$ and then decreases as x is positive.

7. A

Let A be the amplitude,

ω be the angular velocity.

From the graphs:

Maximum displacement = $A = 2$ .... (1)

Maximum velocity $= \omega A = 6$ .... (2)

Maximum acceleration $= \omega^2 A$

From (1) & (2), $\omega = 3$

$\therefore$ $\frac{2\pi}{T} = 3$

$$T = \frac{2\pi}{3}\text{ s}$$

8. D

Amplitude = Half of the greatest difference in displacement

$$= \frac{1}{2}[2 - (-2)]$$

$$= 2\ \mu\text{m}$$

Frequency = $\frac{1}{\text{Period}}$

$$= \frac{1}{20 \times 10^{-6}}$$

$$= 50\text{ kHz}$$

9. B

Self Evaluation Exercise 7.4A (p.266)

1. B

In a SHM of a pendulum,

$$\omega^2 = \frac{g}{\ell}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$$

For $f_1 = \frac{1}{2\pi}\sqrt{\frac{g}{\ell_o}}$, $f_2 = \frac{1}{2\pi}\sqrt{\frac{g}{2\ell_o}} = \frac{1}{\sqrt{2}}f_1$

2. C

From $x = 5\sin(2t)$

$$\omega = 2$$

$$\sqrt{\frac{g}{\ell}} = 2$$

$$\ell = 2.5\text{ m}$$

3. A

In a SHM of a pendulum,

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

In an inertial frame, the motion of the pendulum is the same, so $T_o = T_1$. But if the system is accelerating downwards, which is no longer an inertial frame, g will appear to be smaller.

$\therefore$ $T_o = T_1 < T_2$

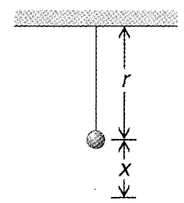
4. E

5. B

6. Let r be the length of the pendulum,

h be the height of the ceiling above the floor,

then $r + x = h$



$$\omega^2 = \frac{g}{r}$$

$$\left(\frac{2\pi}{t}\right)^2 = \frac{g}{r}$$

$$gt^2 = 4\pi^2 r,$$

$$gt^2 = 4\pi^2(h - x)$$

When $t = 3.38$ s, $x = 0.1$ m

$t = 3.20$ s, $x = 0.4$ m

$\therefore$ $\begin{cases} g(3.38)^2 = 4\pi^2(h - 0.1) & \text{.......... (1)} \\ g(3.20)^2 = 4\pi^2(h - 0.4) & \text{.......... (2)} \end{cases}$

(1) – (2), $g[(3.38)^2 - (3.2)^2] = 4\pi^2(-0.1 + 0.4)$

$$g = \frac{4\pi^2(0.3)}{1.1844}$$

$$g = 10\text{ m s}^{-2}$$

$\frac{(1)}{(2)}$: $\frac{(3.38)^2}{(3.2)^2} = \frac{h - 0.1}{h - 0.4}$

$$[(3.38)^2 - (3.2)^2]h = (3.38)^2(0.4) - (3.2)^2(0.1)$$

$$h = 2.99\text{ m}$$

When obtaining the period T of the SHM, the time t of the pendulum to oscillate n times should be taken first, so that the period can be calculated by

$$T = \frac{t}{n}$$

However, it should be noticed that n must be carefully chosen. If n is too small, the percentage error will be very large; if n is too large, the damping effect cannot be neglected for such a long time.

7. **(a)** From the figure,
Amplitude $x_0 = 0.12$ m

(b) From the figure,
Period $T = 2.0$ s

(c) Frequency $f = \frac{1}{T} = \frac{1}{2.0} = 0.50$ Hz

(d) Angular frequency
$\omega = 2\pi f = 2\pi(0.50) = \pi$ rad s^{-1}

(e) **(i)** When displacement $x = 0$,
acceleration $a = 0$

(ii) When $|x| = |x_{max}| = |x_0| = 0.12$ m,
$|a| = \omega^2 x_0 = \pi^2 \times 0.12$
$= 1.18$ m s^{-2}

(f) $|v|$ is maximum when x is zero

$|v_{max}| = \omega\sqrt{x_0^2 - 0} = \pi \times 0.12$

$= 0.38$ m s^{-1}

Self Evaluation Exercise 7.4D (p.270)

1. C

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{20}{0.2}}$$

$$= 10$$

$$a_{max} = \omega^2 A$$

$$3 = (10)^2 A$$

$$A = 0.03\text{ m s}^{-1}$$

$v_{max} = A\omega$
$= 0.03 \times 10$
$= 0.3 \text{ m s}^{-1}$

2. Amplitude of oscillation = b
Angular frequency = ω
Hence, oscillation is described by $x = b\cos(\omega t + \theta_0)$
But $x = b$ when $t = 0 \Rightarrow b = b\cos\theta_0$ or $\theta_0 = 0$
$\therefore\ x = b\cos\omega t$

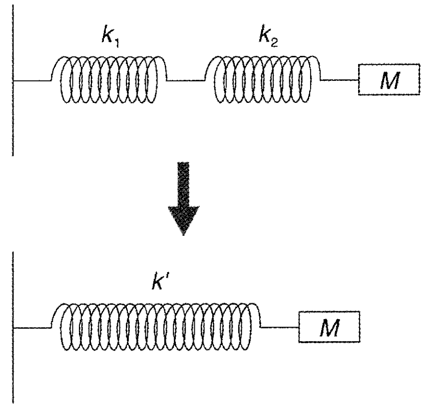
3. HKALE Question

Self Evaluation Exercise 7.4E (p.274)

1. By $F = Ma$

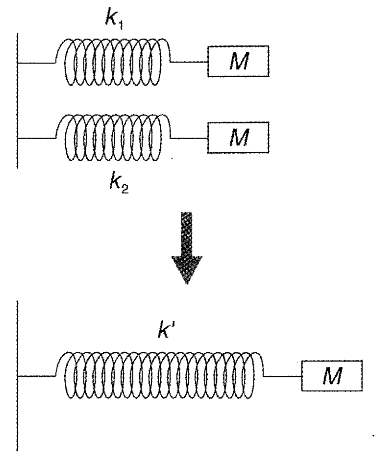
(a)
$$k_1 x = -Ma$$
$$a = -\frac{k_1}{M}x$$
$$\therefore\ \omega = \sqrt{\frac{k_1}{M}}$$
$$T = 2\pi\sqrt{\frac{M}{k_1}}$$

(b) Let k' be the equivalent spring constant.
For a pair of spring connected in series,



$$\frac{1}{k'} = \frac{1}{k_1} + \frac{1}{k_2}$$
$$k' = \frac{k_1 k_2}{k_1 + k_2}$$
$$k'x = -Ma$$
$$a = -\frac{k'}{M}x$$
$$T = 2\pi\sqrt{\frac{M}{k'}} = 2\pi\sqrt{\frac{M(k_1 + k_2)}{k_1 k_2}}$$

(c) For a pair of spring connected in parallel,



$$k' = k_1 + k_2$$
$$k'x = -Ma$$
$$a = -\frac{k'}{M}x$$
$$T = 2\pi\sqrt{\frac{M}{k'}} = 2\pi\sqrt{\frac{M}{k_1 + k_2}}$$

2.
$$F = M\frac{d^2x}{dt^2}$$
$$k(a + x) - k(a - x) = -M\frac{d^2x}{dt^2}$$
$$\frac{d^2x}{dt^2} = -\frac{2k}{M}x$$
$$\therefore\ T = 2\pi\sqrt{\frac{M}{2k}}$$

Self Evaluation Exercise 7.5 (p.278)

1. A

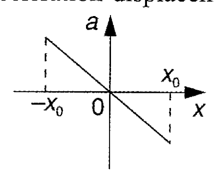
2. C

3. A

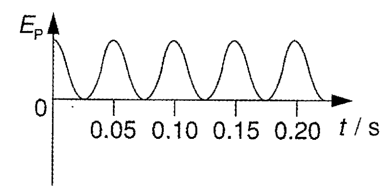
4. C

5. **(a)** **(i)** Simple harmonic motion is defined as oscillatory motion in which the acceleration is proportional, but opposite in direction, to the displacement (from the equilibrium position).

(ii) Sketch acceleration-displacement graph:



(b) Sketch potential energy-time graph:



(c) For the subsequent motion,
The period will be shorter due to a smaller mass of the oscillating system.
The amplitude will be greater due a higher equilibrium position and so a greater initial displacement.

Self Evaluation Exercise 7.6 (p.282)

1. A

From phasor diagrams, v leads x by $\frac{\pi}{2}$ at any time.

2. C
$x = A\sin(\omega t + B)$
$v = \omega A\cos(\omega t + B) = \omega A\sin(\frac{\pi}{2} + \omega t + B)$
$a = -\omega^2 A\sin(\omega t + B) = \omega^2 A\sin(\pi + \omega t + B)$
where A is the amplitude,
B is the initial angular position.
$\therefore$ **(1)** and **(2)** are correct.
(3) is wrong because the displacement and acceleration are 180° out of phase.

3. D
(1) is not correct. In a SHM, the acceleration will change according to $a = -\omega^2 x$.
(2) is correct. In a projectile motion, the only force acting on an object is the gravitational force which is a constant. The object is in constant acceleration and thus its velocity will change.
(3) is correct. If we consider the magnitude only, the acceleration of an object in a uniform circular motion remains unchanged. However, although its speed is also unchanged, the direction keeps changing with time. The velocity is changing.

4. **(a)** Maximum velocity = ωA
$$= \frac{2\pi}{(2 \times 0.05)}\left(\frac{0.01}{2}\right)$$
$$= 0.31 \text{ m s}^{-1}$$

(b) Maximum acceleration = $\omega^2 A$
$$= \left[\frac{2\pi}{(2 \times 0.05)}\right]^2\left(\frac{0.01}{2}\right)$$
$$= 19.74 \text{ m s}^{-2}$$

5.
$x = a\sin(\omega t) = a\sin\theta$
At Z, $x = \frac{1}{2}a = a(\sin\theta)$
$\therefore\ \sin\theta = \frac{1}{2}$
$\theta = \frac{\pi}{6}$
At Y, $\theta = \pi$
$\therefore$ Phase difference = $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

6.
$x = A\sin\omega t$
$v = \omega A\cos\omega t$
When $v = \frac{1}{2}v_{max}$,
$v = \frac{1}{2}\omega A$
$\therefore\ \cos\omega t = \frac{1}{2}$
At that instant t_o, $x = A\sin\omega t_o = A\sqrt{1 - (\cos^2\omega t_o)}$
$$= A\sqrt{1 - \left(\frac{1}{2}\right)^2}$$
$$= A\left(\frac{\sqrt{3}}{2}\right)$$

7.
$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$4(0.5) = 2\pi\sqrt{\frac{0.1}{k}}$$
$$k = \frac{4\pi^2(0.1)}{(2.0)^2}$$
$$= 0.99 \text{ N m}^{-1}$$

Self Evaluation Exercise 7.8 (p.290)

1. E

2. A

Review Exercise 7 (p.298)

A. Multiple Choice Questions

1. D
The velocity v is the slope on the displacement-time graph and the direction of acceleration a is opposite to the displacement. Therefore, a and v are in the same direction when the slope and the displacement are in the opposite directions.

2. A
In the region from M to N, half of the waveform is covered. Moreover, MN is on the time axis, so the physical meaning of MN should also have the same unit as the time axis.

3. A
Initially, $OP = a$
$\therefore$ By $x = a \sin\omega t$

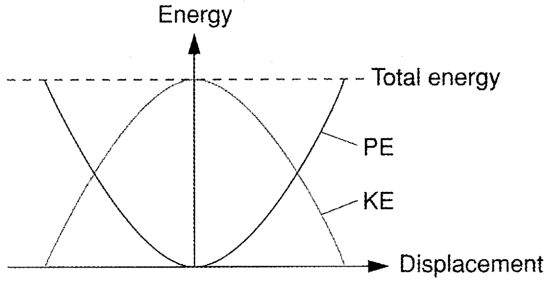
$$\omega t_o = \frac{\pi}{2}$$

After $\frac{5}{8}T = \frac{5}{8}\left(\frac{2\pi}{\omega}\right) = \frac{5\pi}{4\omega}$

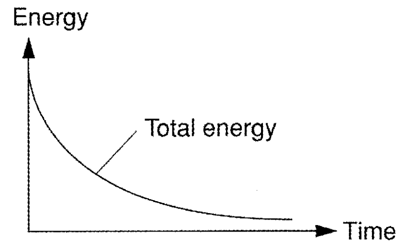
$$x = a\sin\left[\frac{\pi}{2} + \omega\left(\frac{5\pi}{4\omega}\right)\right] = a\sin\left[\frac{7\pi}{4}\right] = a\sin\left(2\pi - \frac{\pi}{4}\right) = -\frac{a}{\sqrt{2}}$$

B. Structured Questions

4. **(a) (i)** In an undamped SHM, the potential energy and the kinetic energy will transform to each other.



(ii) This is because in a slightly damped oscillating system, its energy is decaying exponentially with time.
As total energy = $\frac{1}{2}kA^2$, we will also see its amplitude A decreases with time.



(b) Since the equation of motion of a SHM should have a form of
$$a = -\omega^2 x$$
By comparing it with the given equation, we have
$$b = \omega^2$$
$$b = \frac{4\pi^2}{T^2}$$

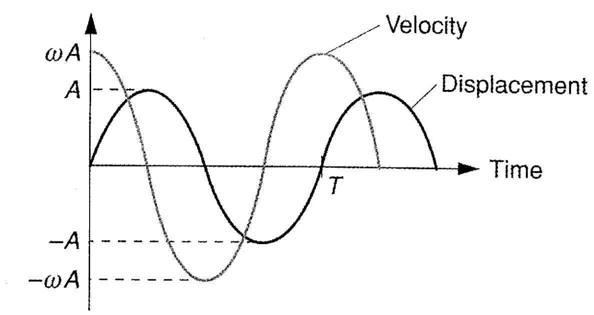
5. For a SHM,
Maximum acceleration = $\omega^2 A$
$$g = \omega^2(0.3)$$
$$T = 2\pi\omega = 2\pi\sqrt{\frac{0.3}{10}} = 1.09 \text{ s}$$
If the diver is heavier, the amplitude A will be greater (> 0.3 m), so T will be greater.

6. **(a)** Yes. The displacement should not exceed the elastic limit or otherwise the spring will not return to the original position.
(b) $ma = -kx$
$$a = -\frac{k}{m}x$$
$$\omega^2 = \frac{k}{m}$$
$$T = 2\pi\sqrt{\frac{m}{k}}$$
When m is doubled, T will be $\sqrt{2}$ times the original period.
(c) There is work done against the air resistance. The kinetic energy is dissipated to the internal energy of air.

7. **(a)** $ma = -kx$
$$a = -\frac{k}{m}x$$
$$\omega = \sqrt{\frac{k}{m}}$$
$$T = 2\pi\sqrt{\frac{m}{k}}$$
(b) (i) In (a), $\frac{1}{k'} = \frac{1}{k} + \frac{1}{k}$
$$k' = \frac{k}{2}$$
$$T = 2\pi\sqrt{\frac{m}{k'}} = \left[2\pi\sqrt{\frac{m}{k}}\right]\sqrt{2}$$
In (b), $k' = k + k = 2k$
$$T = 2\pi\sqrt{\frac{m}{k'}} = \left[2\pi\sqrt{\frac{m}{k}}\right]\left(\frac{1}{\sqrt{2}}\right)$$
(ii) $\frac{T_{series}}{T_{parallel}} = (\sqrt{2}) \div (\frac{1}{\sqrt{2}}) = 2$
(iii) (1) A simple pendulum with water as the medium
(2) Barton's pendulum
(3) Barton's pendulum

8. **(a)** $F = kx$
$0.4 = k(0.01)$
$k = 40 \text{ N m}^{-1}$
$$e = \frac{mg}{k} = \frac{0.2\times 10}{40} = 0.05 \text{ m} = 5 \text{ cm}$$
(b) $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.2}{40}} = 0.44 \text{ s}$
(c) $v_{max} = \omega A = (\frac{2\pi}{T})(0.02) = 0.28 \text{ m s}^{-1}$ when $x = 0$
$a_{max} = \omega^2 A = (\frac{2\pi}{T})^2(0.02) = 4.00 \text{ m s}^{-2}$ when $x = A$
(d) $E = \frac{1}{2}kA^2$
$$= \frac{1}{2}(40)(0.02)^2$$
$$= 8 \times 10^{-3} \text{ J}$$

9. **(a) 1.** The period is independent of the amplitude.
2. The acceleration is proportional to x.
(b)



(c) Set up a pendulum and measure the period starting from small amplitude. Then repeat the experiment for several times with large amplitude.
The period should be independent of amplitude for small angle.
(d) (i) $A = \frac{0.09}{2} = 0.045 \text{ m}$
$F = 3\,000 \text{ min}^{-1} = 50 \text{ Hz}$
Acceleration at the top = $\omega^2 A = (2\pi f)^2 A = 4.44 \times 10^3 \text{ m s}^{-2}$
(ii) Velocity at the mid point = $\pm\,\omega A = \pm\,(2\pi f)A = \pm\,14.14 \text{ m s}^{-1}$

10. **(a)** Displacement x in a SHM is the position of an object at time t from the equilibrium position ($x = A\sin(\omega t + B)$) where A is the amplitude and B is the initial angular position.
(b) The amplitude A of a SHM is the maximum displacement that the object can reach from the equilibrium position.
(c) A complete cycle of SHM can be represented as 2π rad. If there are f cycles per second, the angular frequency is said to be $2\pi f$ rad s^{-1}.

C. Overseas & HKALE Questions

11. **(a)** Every oscillating system has its own natural frequency of oscillation.
Resonance refers to the phenomenon that the amplitude of oscillations reaches a maximum when the frequency of the driving oscillations is equal to the natural frequency of the driven oscillating system.
(b) (i) Frequency of the water waves,
$$f_w = \frac{\text{Speed}}{\text{Wavelength}} = \frac{0.90}{0.30} = 3.0 \text{ Hz}$$
(ii) Resonance occurs
$\Rightarrow$ Natural frequency of block $f = f_w = 3.0$ Hz
$$\frac{1}{2\pi}\sqrt{\frac{28}{m}} = f$$

$$\Rightarrow \quad 3.0 = \frac{1}{2\pi}\sqrt{\frac{28}{m}}$$

∴ Mass of the block,
$m = 7.88 \times 10^{-2}$ kg

(c) **(i)** Water waves of larger amplitude will increase the rate of transfer of energy which will increase the amplitude of oscillations of the block.

(ii) With increase in the distance between wave crests, corresponding to an increase in wavelength and so a decrease in frequency, there would no longer be resonance and the amplitude of oscillations of the block would decrease.

(iii) When the block has absorbed water, its mass m is increased and the block's natural frequency is changed. There would no longer be resonance and the amplitude of oscillations of the block would decrease.

12. **(a)** The phenomenon illustrated in Fig. (b) is known as resonance.

(b) From Fig. (b), at maximum amplitude, frequency $f = 12.5$ Hz

(i) Angular frequency,
$\omega = 2\pi f$
$= 2\pi(12.5)$
$= 78.54$ rad s^{-1}

(ii) Period
$$T = \frac{1}{f} = \frac{1}{12.5} = 0.08 \text{ s}$$

(c) The light card will increase the degree of damping on the system.
Draw line (----) with a lower peak and occurring at a slightly lower frequency.

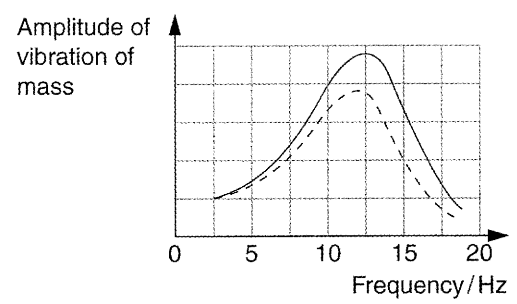


Fig. (b)

(d) In musical instruments, when the vibration of a reed or string matches the natural frequency of an air column, resonance occurs and a loud note is produced.

13. **(a)** **(i)** From Fig. (b)
the period, $T = 0.60$ s

(ii) The angular frequency,
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.60} = 10.47 \text{ rad s}^{-1}$$

(b) **(i)** From Fig. (b) and Fig. (c),
the time interval $\Delta t = 0.20$ s

(ii) The phase angle
$$\phi = \frac{\Delta t}{T} \times 2\pi = \frac{0.20 \times 2\pi}{0.60} = 2.09 \text{ rad}$$

(c) **(i)** Damping refers to the reduction of the amplitude of an oscillating system due to loss of energy caused by dissipative forces such as friction and viscosity.

(ii) **1.** Light damping refers to a gradual reduction of the amplitude of oscillation. For the given spring and mass system, light damping may be achieved by letting the mass oscillate in water.

2. The degree of damping may be increased by letting the mass oscillates in a more viscous fluid such as oil.

14. – 16. HKALE Questions